

# Linear Programming

# B

MODULE

## MODULE OUTLINE

- ◆ Why Use Linear Programming? 700
- ◆ Requirements of a Linear Programming Problem 701
- ◆ Formulating Linear Programming Problems 701
- ◆ Graphical Solution to a Linear Programming Problem 702
- ◆ Sensitivity Analysis 705
- ◆ Solving Minimization Problems 708
- ◆ Linear Programming Applications 710
- ◆ The Simplex Method of LP 713
- ◆ Integer and Binary Variables 713



Alaska Airlines



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# LEARNING OBJECTIVES

- LO B.1** *Formulate* linear programming models, including an objective function and constraints 702
- LO B.2** *Graphically* solve an LP problem with the iso-profit line method 704
- LO B.3** *Graphically* solve an LP problem with the corner-point method 705
- LO B.4** *Interpret* sensitivity analysis and shadow prices 706
- LO B.5** *Construct* and solve a minimization problem 709
- LO B.6** *Formulate* production-mix, diet, and labor scheduling problems 710

The storm front closed in quickly on Boston's Logan Airport, shutting it down without warning. The heavy snowstorms and poor visibility sent airline passengers and ground crew scurrying. Because airlines use linear programming (LP) to schedule flights, hotels, crews, and refueling, LP has a direct impact on profitability. If an airline gets a major weather disruption at one of its hubs, a lot of flights may get canceled, which means a lot of crews and airplanes in the wrong places. LP is the tool that helps airlines unspool and cope with this weather mess.



Paul Italiano/Alamy

## Why Use Linear Programming?

Many operations management decisions involve trying to make the most effective use of an organization's resources. Resources typically include machinery (such as planes, in the case of an airline), labor (such as pilots), money, time, and raw materials (such as jet fuel). These resources may be used to produce products (such as machines, furniture, food, or clothing) or services (such as airline schedules, advertising policies, or investment decisions). **Linear programming (LP)** is a widely used mathematical technique designed to help operations managers plan and make the decisions necessary to allocate resources.

A few examples of problems in which LP has been successfully applied in operations management are:

1. Scheduling school buses to *minimize* the total distance traveled when carrying students
2. Allocating police patrol units to high crime areas to *minimize* response time to 911 calls
3. Scheduling tellers at banks so that needs are met during each hour of the day while *minimizing* the total cost of labor
4. Selecting the product mix in a factory to make best use of machine- and labor-hours available while *maximizing* the firm's profit
5. Picking blends of raw materials in feed mills to produce finished feed combinations at *minimum* cost
6. Determining the distribution system that will *minimize* total shipping cost from several warehouses to various market locations
7. Developing a production schedule that will satisfy future demands for a firm's product and at the same time *minimize* total production and inventory costs
8. Allocating space for a tenant mix in a new shopping mall to *maximize* revenues to the leasing company

### Linear programming (LP)

A mathematical technique designed to help operations managers plan and make decisions necessary to allocate resources.

#### VIDEO B.1

Scheduling Challenges at Alaska Airlines

# Requirements of a Linear Programming Problem

All LP problems have four requirements: an objective, constraints, alternatives, and linearity:

1. LP problems seek to *maximize* or *minimize* some quantity (usually profit or cost). We refer to this property as the **objective function** of an LP problem. The major objective of a typical firm is to maximize dollar profits in the long run. In the case of a trucking or air-line distribution system, the objective might be to minimize shipping costs.
2. The presence of restrictions, or **constraints**, limits the degree to which we can pursue our objective. For example, deciding how many units of each product in a firm’s product line to manufacture is restricted by available labor and machinery. We want, therefore, to maximize or minimize a quantity (the objective function) subject to limited resources (the constraints).
3. There must be *alternative courses of action* to choose from. For example, if a company produces three different products, management may use LP to decide how to allocate among them its limited production resources (of labor, machinery, and so on). If there were no alternatives to select from, we would not need LP.
4. The objective and constraints in linear programming problems must be expressed in terms of *linear equations* or inequalities. Linearity implies proportionality and additivity. If  $x_1$  and  $x_2$  are decision variables, there can be no products (e.g.,  $x_1x_2$ ) or powers (e.g.,  $x_1^3$ ) in the objective or constraints. For example, the expression  $5x_1 + 8x_2 \leq 250$  is linear; however, the expression  $5x_1 + 8x_2 - 2x_1x_2 \leq 300$  is not linear.

### Objective function

A mathematical expression in linear programming that maximizes or minimizes some quantity (often profit or cost, but any goal may be used).

### Constraints

Restrictions that limit the degree to which a manager can pursue an objective.

# Formulating Linear Programming Problems

One of the most common linear programming applications is the *product-mix problem*. Two or more products are usually produced using limited resources. The company would like to determine how many units of each product it should produce to maximize overall profit given its limited resources. Let’s look at an example.

### STUDENT TIP

Here we set up an LP example that we will follow for most of this module.

## Glickman Electronics Example

The Glickman Electronics Company in Washington, DC, produces two products: (1) the Glickman x-pod and (2) the Glickman BlueBerry. The production process for each product is similar in that both require a certain number of hours of electronic work and a certain number of labor-hours in the assembly department. Each x-pod takes 4 hours of electronic work and 2 hours in the assembly shop. Each BlueBerry requires 3 hours in electronics and 1 hour in assembly. During the current production period, 240 hours of electronic time are available, and 100 hours of assembly department time are available. Each x-pod sold yields a profit of \$7; each BlueBerry produced may be sold for a \$5 profit.

Glickman’s problem is to determine the best possible combination of x-pods and BlueBerrys to manufacture to reach the maximum profit. This product-mix situation can be formulated as a linear programming problem.

We begin by summarizing the information needed to formulate and solve this problem (see Table B.1). Further, let’s introduce some simple notation for use in the objective function and constraints. Let:

$X_1$  = number of x-pods to be produced

$X_2$  = number of BlueBerrys to be produced

### ACTIVE MODEL B.1

This example is further illustrated in Active Model B.1 in MyOMLab.

TABLE B.1 Glickman Electronics Company Problem Data

DEPARTMENT	HOURS REQUIRED TO PRODUCE ONE UNIT		AVAILABLE HOURS THIS WEEK
	X-PODS ( $X_1$ )	BLUEBERRYS ( $X_2$ )	
Electronic	4	3	240
Assembly	2	1	100
Profit per unit	\$7	\$5	

Now we can create the LP *objective function* in terms of  $X_1$  and  $X_2$ :

$$\text{Maximize profit} = \$7X_1 + \$5X_2$$

Our next step is to develop mathematical relationships to describe the two constraints in this problem. One general relationship is that the amount of a resource used is to be less than or equal to ( $\leq$ ) the amount of resource *available*.

**LO B.1** Formulate linear programming models, including an objective function and constraints

*First constraint:* Electronic time used is  $\leq$  Electronic time available.

$$4X_1 + 3X_2 \leq 240 \text{ (hours of electronic time)}$$

*Second constraint:* Assembly time used is  $\leq$  Assembly time available.

$$2X_1 + 1X_2 \leq 100 \text{ (hours of assembly time)}$$

Both these constraints represent production capacity restrictions and, of course, affect the total profit. For example, Glickman Electronics cannot produce 70 x-pods during the production period because if  $X_1 = 70$ , both constraints will be violated. It also cannot make  $X_1 = 50$  x-pods and  $X_2 = 10$  BlueBerrys. This constraint brings out another important aspect of linear programming; that is, certain interactions will exist between variables. The more units of one product that a firm produces, the fewer it can make of other products.

## Graphical Solution to a Linear Programming Problem

The easiest way to solve a small LP problem such as that of the Glickman Electronics Company is the **graphical solution approach**. The graphical procedure can be used only when there are two **decision variables** (such as number of x-pods to produce,  $X_1$ , and number of BlueBerrys to produce,  $X_2$ ). When there are more than two variables, it is *not* possible to plot the solution on a two-dimensional graph; we then must turn to more complex approaches described later in this module.

### Graphical solution approach

A means of plotting a solution to a two-variable problem on a graph.

### Decision variables

Choices available to a decision maker.

## Graphical Representation of Constraints

To find the optimal solution to a linear programming problem, we must first identify a set, or region, of feasible solutions. The first step in doing so is to plot the problem's constraints on a graph.

The variable  $X_1$  (x-pods, in our example) is usually plotted as the horizontal axis of the graph, and the variable  $X_2$  (BlueBerrys) is plotted as the vertical axis. The complete problem may be restated as:

$$\text{Maximize profit} = \$7X_1 + \$5X_2$$

Subject to the constraints:

$$4X_1 + 3X_2 \leq 240 \text{ (electronics constraint)}$$

$$2X_1 + 1X_2 \leq 100 \text{ (assembly constraint)}$$

$$X_1 \geq 0 \text{ (number of x-pods produced is greater than or equal to 0)}$$

$$X_2 \geq 0 \text{ (number of BlueBerrys produced is greater than or equal to 0)}$$

(These last two constraints are also called *nonnegativity constraints*.)

The first step in graphing the constraints of the problem is to convert the constraint *inequalities* into *equalities* (or equations):

$$\text{Constraint A: } 4X_1 + 3X_2 = 240$$

$$\text{Constraint B: } 2X_1 + 1X_2 = 100$$

The equation for constraint A is plotted in Figure B.1 and for constraint B in Figure B.2.

To plot the line in Figure B.1, all we need to do is to find the points at which the line  $4X_1 + 3X_2 = 240$  intersects the  $X_1$  and  $X_2$  axes. When  $X_1 = 0$  (the location where the line touches the  $X_2$  axis), it implies that  $3X_2 = 240$  and that  $X_2 = 80$ . Likewise, when  $X_2 = 0$ , we see that  $4X_1 = 240$  and that  $X_1 = 60$ . Thus, constraint A is bounded by the line running from ( $X_1 = 0, X_2 = 80$ ) to ( $X_1 = 60, X_2 = 0$ ). The shaded area represents all points that satisfy the original *inequality*.

Constraint B is illustrated similarly in Figure B.2. When  $X_1 = 0$ , then  $X_2 = 100$ ; and when  $X_2 = 0$ , then  $X_1 = 50$ . Constraint B, then, is bounded by the line between

### STUDENT TIP

We named the decision variables  $X_1$  and  $X_2$  here, but any notations (e.g.,  $x$ - $p$  and  $B$  or  $X$  and  $Y$ ) would do as well.

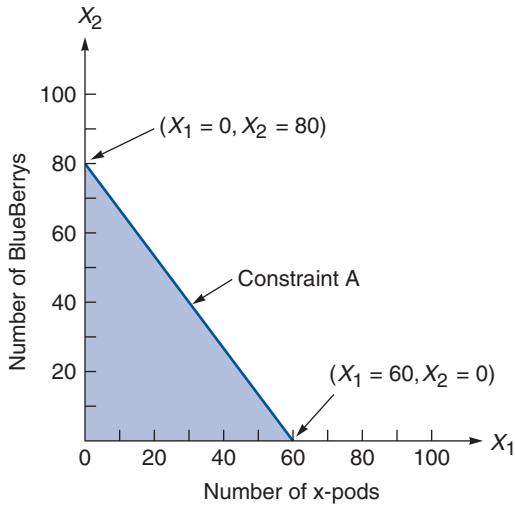


Figure B.1

**Constraint A**

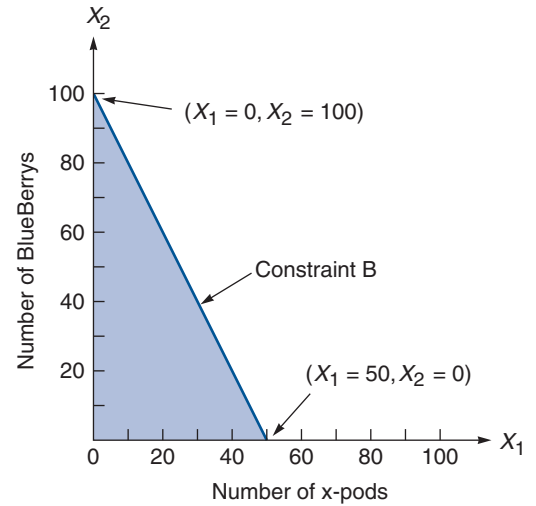


Figure B.2

**Constraint B**

$(X_1 = 0, X_2 = 100)$  and  $(X_1 = 50, X_2 = 0)$ . The shaded area represents the original inequality.

Figure B.3 shows both constraints together (along with the nonnegativity constraints). The shaded region is the part that satisfies all restrictions. The shaded region in Figure B.3 is called the *area of feasible solutions*, or simply the **feasible region**. This region must satisfy *all* conditions specified by the program's constraints and is thus the region where all constraints overlap. Any point in the region would be a *feasible solution* to the Glickman Electronics Company problem. Any point outside the shaded area would represent an *infeasible solution*. Hence, it would be feasible to manufacture 30 x-pods and 20 BlueBerrys  $(X_1 = 30, X_2 = 20)$ , but it would violate the constraints to produce 70 x-pods and 40 BlueBerrys. This can be seen by plotting these points on the graph of Figure B.3.

**Feasible region**

The set of all feasible combinations of decision variables.

**Iso-Profit Line Solution Method**

Now that the feasible region has been graphed, we can proceed to find the *optimal* solution to the problem. The optimal solution is the point lying in the feasible region that produces the highest profit.

Once the feasible region has been established, several approaches can be taken in solving for the optimal solution. The speediest one to apply is called the **iso-profit line method**.<sup>1</sup>

We start by letting profits equal some arbitrary but small dollar amount. For the Glickman Electronics problem, we may choose a profit of \$210. This is a profit level that can easily be obtained without violating either of the two constraints. The objective function can be written as  $\$210 = 7X_1 + 5X_2$ .

This expression is just the equation of a line; we call it an *iso-profit line*. It represents all combinations (of  $X_1, X_2$ ) that will yield a total profit of \$210. To plot the profit line, we proceed exactly as we did to plot a constraint line. First, let  $X_1 = 0$  and solve for the point at which the line crosses the  $X_2$  axis:

$$\begin{aligned} \$210 &= \$7(0) + \$5X_2 \\ X_2 &= 42 \text{ BlueBerrys} \end{aligned}$$

Then let  $X_2 = 0$  and solve for  $X_1$ :

$$\begin{aligned} \$210 &= \$7X_1 + \$5(0) \\ X_1 &= 30 \text{ x-pods} \end{aligned}$$

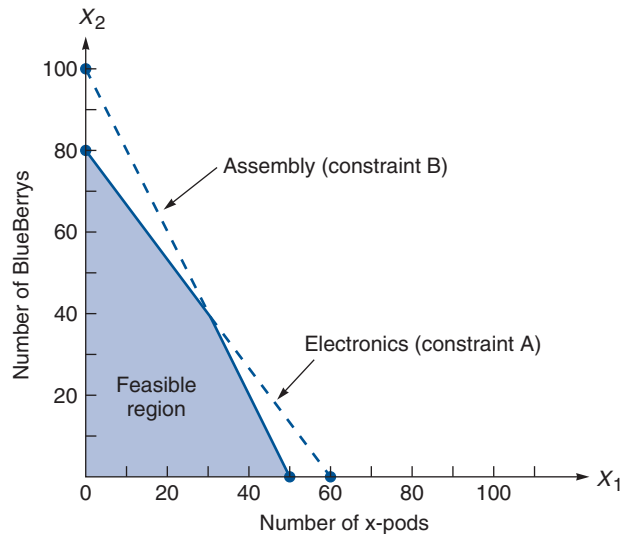


Figure B.3

**Feasible Solution Region for the Glickman Electronics Company Problem**

**Iso-profit line method**

An approach to solving a linear programming maximization problem graphically.

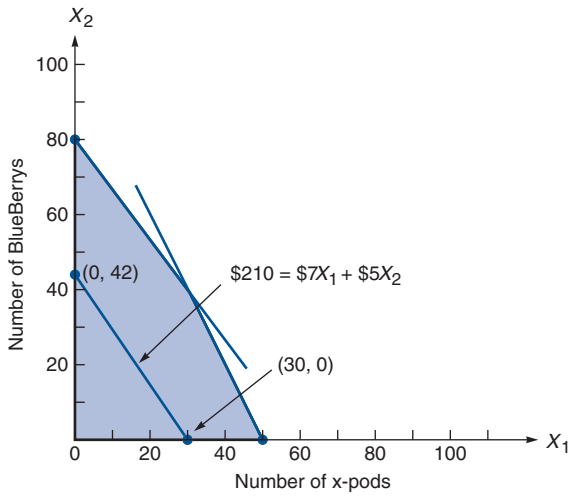


Figure B.4

A Profit Line of \$210 Plotted for the Glickman Electronics Company

**LO B.2** Graphically solve an LP problem with the iso-profit line method

We can now connect these two points with a straight line. This profit line is illustrated in Figure B.4. All points on the line represent feasible solutions that produce a profit of \$210.

We see, however, that the iso-profit line for \$210 does not produce the highest possible profit to the firm. In Figure B.5, we try graphing three more lines, each yielding a higher profit. The middle equation,  $\$280 = \$7X_1 + \$5X_2$ , was plotted in the same fashion as the lower line. When  $X_1 = 0$ :

$$\begin{aligned} \$280 &= \$7(0) + \$5X_2 \\ X_2 &= 56 \text{ BlueBerrys} \end{aligned}$$

When  $X_2 = 0$ :

$$\begin{aligned} \$280 &= \$7X_1 + \$5(0) \\ X_1 &= 40 \text{ x-pods} \end{aligned}$$

Again, any combination of x-pods ( $X_1$ ) and BlueBerrys ( $X_2$ ) on this iso-profit line will produce a total profit of \$280.

Note that the third line generates a profit of \$350, even more of an improvement. The farther we move from the 0 origin, the higher our profit will be. Another important point to note is that these iso-profit lines are parallel. We now have two clues how to find the optimal solution to the original problem. We can draw a series of parallel profit lines (by carefully moving our ruler in a plane parallel to the first profit line). The highest profit line that still touches some point of the feasible region will pinpoint the optimal solution. Notice that the fourth line (\$420) is too high to count because it does not touch the feasible region.

The highest possible iso-profit line is illustrated in Figure B.6. It touches the tip of the feasible region at the point where the two resource constraints intersect. To find its coordinates *accurately*, we will have to solve for the intersection of the two constraint lines. As you may recall from algebra, we can apply the method of *simultaneous equations* to the two constraint equations:

$$\begin{aligned} 4X_1 + 3X_2 &= 240 && (\text{electronics time}) \\ 2X_1 + 1X_2 &= 100 && (\text{assembly time}) \end{aligned}$$

To solve these equations simultaneously, we multiply the second equation by  $-2$ :

$$-2(2X_1 + 1X_2 = 100) = -4X_1 - 2X_2 = -200$$

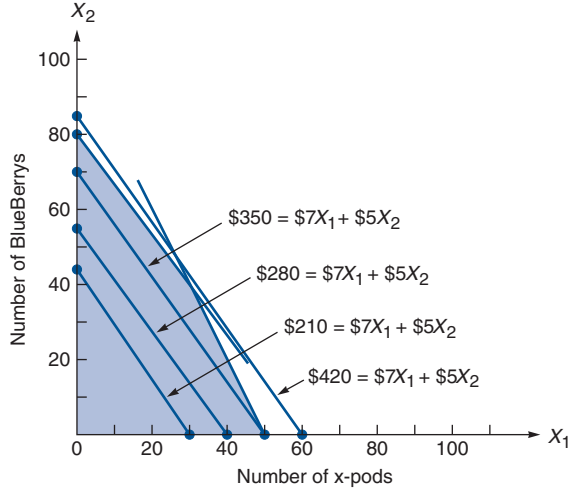


Figure B.5

Four Iso-Profit Lines Plotted for the Glickman Electronics Company

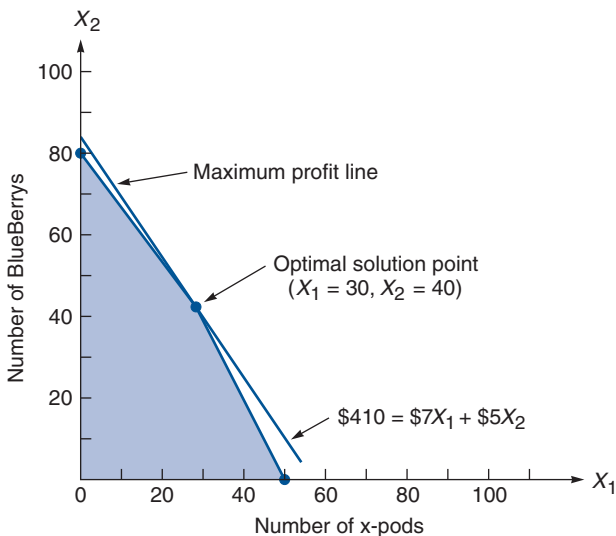


Figure B.6

Optimal Solution for the Glickman Electronics Problem

and then add it to the first equation:

$$\begin{array}{r} +4X_1 + 3X_2 = 240 \\ -4X_1 - 2X_2 = -200 \\ \hline + 1X_2 = 40 \end{array}$$

or:

$$X_2 = 40$$

Doing this has enabled us to eliminate one variable,  $X_1$ , and to solve for  $X_2$ . We can now substitute 40 for  $X_2$  in either of the original constraint equations and solve for  $X_1$ . Let us use the first equation. When  $X_2 = 40$ , then:

$$\begin{array}{r} 4X_1 + 3(40) = 240 \\ 4X_1 + 120 = 240 \\ 4X_1 = 120 \\ X_1 = 30 \end{array}$$

Thus, the optimal solution has the coordinates  $(X_1 = 30, X_2 = 40)$ . The profit at this point is  $\$7(30) + \$5(40) = \$410$ .

### Corner-Point Solution Method

A second approach to solving linear programming problems employs the **corner-point method**. This technique is simpler in concept than the iso-profit line approach, but it involves looking at the profit at every corner point of the feasible region.

#### Corner-point method

A method for solving graphical linear programming problems.

The mathematical theory behind linear programming states that an optimal solution to any problem (that is, the values of  $X_1, X_2$  that yield the maximum profit) will lie at a *corner point*, or *extreme point*, of the feasible region. Hence, it is necessary to find only the values of the variables at each corner; the maximum profit or optimal solution will lie at one (or more) of them.

Once again we can see (in Figure B.7) that the feasible region for the Glickman Electronics Company problem is a four-sided polygon with four corner, or extreme, points. These points are labeled ①, ②, ③, and ④ on the graph. To find the  $(X_1, X_2)$  values producing the maximum profit, we find out what the coordinates of each corner point are, then determine and compare their profit levels. (We showed how to find the coordinates for point ③ in the previous section describing the iso-profit line solution method.)

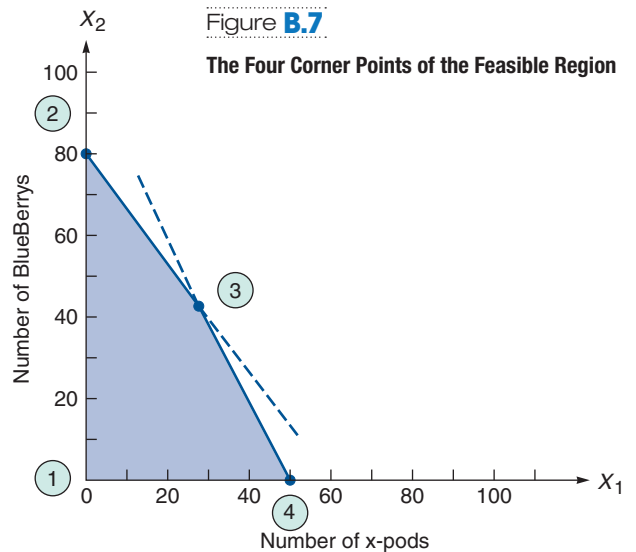


Figure B.7

The Four Corner Points of the Feasible Region

- Point ①:  $(X_1 = 0, X_2 = 0)$  Profit  $\$7(0) + \$5(0) = \$0$
- Point ②:  $(X_1 = 0, X_2 = 80)$  Profit  $\$7(0) + \$5(80) = \$400$
- Point ③:  $(X_1 = 30, X_2 = 40)$  Profit  $\$7(30) + \$5(40) = \$410$
- Point ④:  $(X_1 = 50, X_2 = 0)$  Profit  $\$7(50) + \$5(0) = \$350$

Because point ③ produces the highest profit of any corner point, the product mix of  $X_1 = 30$  x-pods and  $X_2 = 40$  BlueBerrys is the optimal solution to the Glickman Electronics problem. This solution will yield a profit of \$410 per production period; it is the same solution we obtained using the iso-profit line method.

**LO B.3** Graphically solve an LP problem with the corner-point method

## Sensitivity Analysis

Operations managers are usually interested in more than the optimal solution to an LP problem. In addition to knowing the value of each decision variable (the  $X_i$ s) and the value of the objective function, they want to know how sensitive these answers are to input **parameter** changes. For example, what happens if the coefficients of the objective function are not exact, or if they change by 10% or 15%? What happens if the right-hand-side values of the constraints

#### Parameter

Numerical value that is given in a model.

**Sensitivity analysis**

An analysis that projects how much a solution may change if there are changes in the variables or input data.

change? Because solutions are based on the assumption that input parameters are constant, the subject of sensitivity analysis comes into play. **Sensitivity analysis**, or postoptimality analysis, is the study of how sensitive solutions are to parameter changes.

There are two approaches to determining just how sensitive an optimal solution is to changes. The first is simply a trial-and-error approach. This approach usually involves resolving the entire problem, preferably by computer, each time one input data item or parameter is changed. It can take a long time to test a series of possible changes in this way.

The approach we prefer is the analytic postoptimality method. After an LP problem has been solved, we determine a range of changes in problem parameters that will not affect the optimal solution or change the variables in the solution. This is done without resolving the whole problem. LP software, such as Excel's Solver or POM for Windows, has this capability. Let us examine several scenarios relating to the Glickman Electronics example.

**STUDENT TIP** ⚡

Here we look at the sensitivity of the final answers to changing inputs.

Program B.1 is part of the Excel Solver computer-generated output available to help a decision maker know whether a solution is relatively insensitive to reasonable changes in one or more of the parameters of the problem. (The complete computer run for these data, including input and full output, is illustrated in Programs B.3 and B.4 later in this module.)

**LO B.4** Interpret sensitivity analysis and shadow prices

**Sensitivity Report**

The Excel *Sensitivity Report* for the Glickman Electronics example in Program B.1 has two distinct components: (1) a table titled **Variable Cells** and (2) a table titled **Constraints**. These tables permit us to answer several what-if questions regarding the problem solution.

It is important to note that while using the information in the sensitivity report to answer what-if questions, we assume that we are considering a change to only a *single* input data value at a time. That is, the sensitivity information does not always apply to simultaneous changes in several input data values.

The *Variable Cells* table presents information regarding the impact of changes to the objective function coefficients (i.e., the unit profits of \$7 and \$5) on the optimal solution. The *Constraints* table presents information related to the impact of changes in constraint right-hand-side (RHS) values (i.e., the 240 hours and 100 hours) on the optimal solution. Although different LP software packages may format and present these tables differently, the programs all provide essentially the same information.

**Changes in the Resources or Right-Hand-Side Values**

The right-hand-side values of the constraints often represent resources available to the firm. The resources could be labor-hours or machine time or perhaps money or production materials available. In the Glickman Electronics example, the two resources are hours available of

**Program B.1**

**Sensitivity Analysis for Glickman Electronics, Using Excel's Solver**

**Microsoft Excel 15.0 Sensitivity Report**  
Report Created: 9:22:18 AM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$5	Variable Values x-pods	30	0	7	3	0.333333333
\$C\$5	Variable Values BlueBerrys	40	0	5	0.25	1.5

The solution values for the variables appear. We should make 30 x-pods and 40 BlueBerrys.

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$8	Electronic Time Available	240	1.5	240	60	40
\$D\$9	Assembly Time Available	100	0.5	100	20	20

We will use 240 hours and 100 hours of Electronics and Assembly time, respectively.

If we use 1 more Electronics hour, our profit will increase by \$1.50. This is true for up to 60 more hours. The profit will fall by \$1.50 for each Electronics hour less than 240 hours, down to as low as 200 hours.



electronics time and hours of assembly time. If additional hours were available, a higher total profit could be realized. How much should the company be willing to pay for additional hours? Is it profitable to have some additional electronics hours? Should we be willing to pay for more assembly time? Sensitivity analysis about these resources will help us answer these questions.

If the right-hand side of a constraint is changed, the feasible region will change (unless the constraint is redundant), and often the optimal solution will change. In the Glickman example, there were 100 hours of assembly time available each week and the maximum possible profit was \$410. If the available assembly hours are *increased* to 110 hours, the new optimal solution seen in Figure B.8(a) is (45,20) and the profit is \$415. Thus, the extra 10 hours of time resulted in an increase in profit of \$5 or \$0.50 per hour. If the hours are *decreased* to 90 hours as shown in Figure B.8(b), the new optimal solution is (15,60) and the profit is \$405. Thus, reducing the hours by 10 results in a decrease in profit of \$5 or \$0.50 per hour. This \$0.50 per hour change in profit that resulted from a change in the hours available is called the shadow price, or **dual value**. The **shadow price** for a constraint is the improvement in the objective function value that results from a one-unit increase in the right-hand side of the constraint.

**Shadow price (or dual value)**

The value of one additional unit of a scarce resource in LP.

**Validity Range for the Shadow Price** Given that Glickman Electronics’ profit increases by \$0.50 for each additional hour of assembly time, does it mean that Glickman can do this indefinitely, essentially earning infinite profit? Clearly, this is illogical. How far can Glickman increase its assembly time availability and still earn an extra \$0.50 profit per hour? That is, for what level of increase in the RHS value of the assembly time constraint is the shadow price of \$0.50 valid?

The shadow price of \$0.50 is valid as long as the available assembly time stays in a range within which all current corner points continue to exist. The information to compute the upper and lower limits of this range is given by the entries labeled Allowable Increase and Allowable Decrease in the *Sensitivity Report* in Program B.1. In Glickman’s case, these values show that the shadow price of \$0.50 for assembly time availability is valid for an increase of up to 20 hours from the current value and a decrease of up to 20 hours. That is, the available assembly time can range from a low of 80 (= 100 – 20) to a high of 120 (= 100 + 20) for the shadow price of \$0.50 to be valid. Note that the allowable decrease implies that for each hour of assembly time that Glickman loses (up to 20 hours), its profit decreases by \$0.50.

**Changes in the Objective Function Coefficient**

Let us now focus on the information provided in Program B.1 titled **Variable Cells**. Each row in the **Variable Cells** table contains information regarding a decision variable (i.e., x-pods or BlueBerrys) in the LP model.

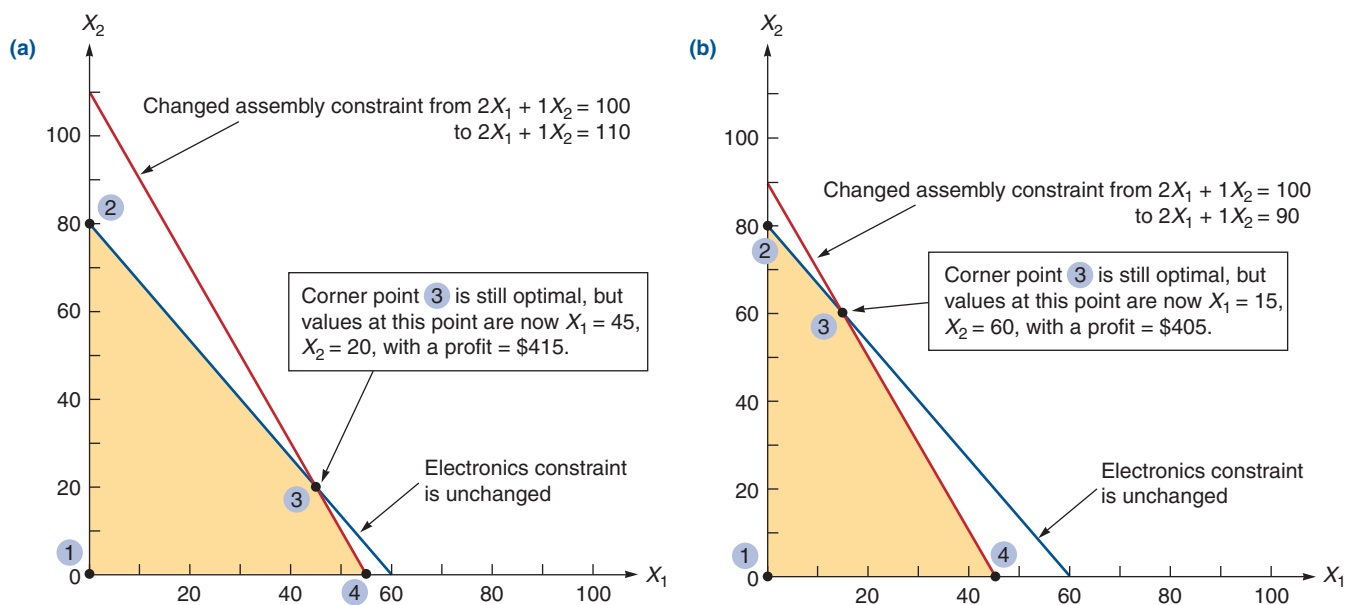


Figure B.8

Glickman Electronics Sensitivity Analysis on the Right-Hand Side (RHS) of the Assembly Resource Constraint

**Allowable Ranges for Objective Function Coefficients** As the unit profit contribution of either product changes, the slope of the iso-profit lines we saw earlier in Figure B.5 changes. The size of the feasible region, however, remains the same. That is, the locations of the corner points do not change.

The limits to which the profit coefficient of x-pods or BlueBerrys can be changed without affecting the optimality of the current solution is revealed by the values in the **Allowable Increase** and **Allowable Decrease** columns of the *Sensitivity Report* in Program B.1. The allowable increase in the objective function coefficient for BlueBerrys is only \$0.25. In contrast, the allowable decrease is \$1.50. Hence, if the unit profit of BlueBerrys drops to \$4 (i.e., a decrease of \$1 from the current value of \$5), it is still optimal to produce 30 x-pods and 40 BlueBerrys. The total profit will drop to \$370 (from \$410) because each BlueBerry now yields less profit (of \$1 per unit). However, if the unit profit drops below \$3.50 per BlueBerry (i.e., a decrease of more than \$1.50 from the current \$5 profit), the current solution is no longer optimal. The LP problem will then have to be resolved using Solver, or other software, to find the new optimal corner point.

**STUDENT TIP** ◆

LP problems can be structured to minimize costs as well as maximize profits.

## Solving Minimization Problems

Many linear programming problems involve *minimizing* an objective such as cost instead of maximizing a profit function. A restaurant, for example, may wish to develop a work schedule to meet staffing needs while minimizing the total number of employees. Also, a manufacturer may seek to distribute its products from several factories to its many regional warehouses in a way that minimizes total shipping costs.

Minimization problems can be solved graphically by first setting up the feasible solution region and then using either the corner-point method or an **iso-cost** line approach (which is analogous to the iso-profit approach in maximization problems) to find the values of  $X_1$  and  $X_2$  that yield the minimum cost.

Example B1 shows how to solve a minimization problem.

**Iso-cost**

An approach to solving a linear programming minimization problem graphically.

### Example B1

#### A MINIMIZATION PROBLEM WITH TWO VARIABLES

Cohen Chemicals, Inc., produces two types of photo-developing fluids. The first, a black-and-white picture chemical, costs Cohen \$2,500 per ton to produce. The second, a color photo chemical, costs \$3,000 per ton.

Based on an analysis of current inventory levels and outstanding orders, Cohen's production manager has specified that at least 30 tons of the black-and-white chemical and at least 20 tons of the color chemical must be produced during the next month. In addition, the manager notes that an existing inventory of a highly perishable raw material needed in both chemicals must be used within 30 days. To avoid wasting the expensive raw material, Cohen must produce a total of at least 60 tons of the photo chemicals in the next month.

**APPROACH** ► Formulate this information as a minimization LP problem.

Let:

$X_1$  = number of tons of black-and-white photo chemical produced

$X_2$  = number of tons of color photo chemical produced

Objective: Minimize cost =  $\$2,500X_1 + \$3,000X_2$

Subject to:

$X_1 \geq 30$  tons of black-and-white chemical

$X_2 \geq 20$  tons of color chemical

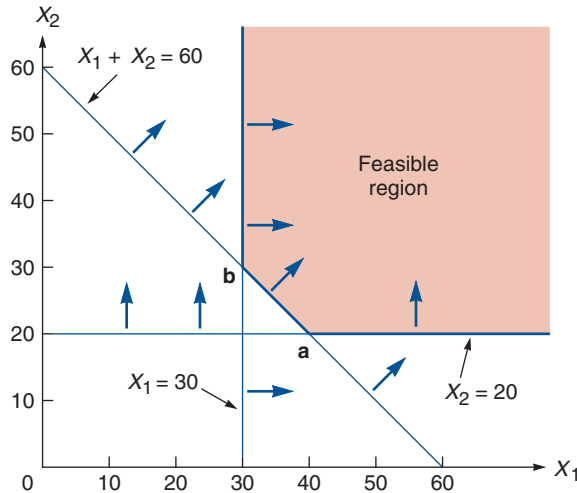
$X_1 + X_2 \geq 60$  tons total

$X_1, X_2 \geq 0$  nonnegativity requirements

**SOLUTION** ► To solve the Cohen Chemicals problem graphically, we construct the problem's feasible region, shown in Figure B.9.

Figure B.9

Cohen Chemicals' Feasible Region



**LO B.5** Construct and solve a minimization problem

Minimization problems are often unbounded outward (that is, on the right side and on the top), but this characteristic causes no problem in solving them. As long as they are bounded inward (on the left side and the bottom), we can establish corner points. The optimal solution will lie at one of the corners.

In this case, there are only two corner points, **a** and **b**, in Figure B.9. It is easy to determine that at point **a**,  $X_1 = 40$  and  $X_2 = 20$ , and that at point **b**,  $X_1 = 30$  and  $X_2 = 30$ . The optimal solution is found at the point yielding the lowest total cost.

Thus:

$$\begin{aligned} \text{Total cost at } \mathbf{a} &= 2,500X_1 + 3,000X_2 \\ &= 2,500(40) + 3,000(20) \\ &= \$160,000 \end{aligned}$$

$$\begin{aligned} \text{Total cost at } \mathbf{b} &= 2,500X_1 + 3,000X_2 \\ &= 2,500(30) + 3,000(30) \\ &= \$165,000 \end{aligned}$$

The lowest cost to Cohen Chemicals is at point **a**. Hence the operations manager should produce 40 tons of the black-and-white chemical and 20 tons of the color chemical.

**INSIGHT** ► The area is either not bounded to the right or above in a minimization problem (as it is in a maximization problem).

**LEARNING EXERCISE** ► Cohen's second constraint is recomputed and should be  $X_2 \geq 15$ . Does anything change in the answer? [Answer: Now  $X_1 = 45$ ,  $X_2 = 15$ , and total cost = \$157,500.]

**RELATED PROBLEMS** ► B.25–B.31 (B.32, B.33 are available in MyOMLab)

**EXCEL OM** Data File ModBExB1.xls can be found in MyOMLab.

## OM in Action

### LP at UPS

On an *average* day, the \$58.2 billion shipping giant UPS delivers 18 million packages to 8.2 million customers in 220 countries. On a *really* busy day, say a few days before Christmas, it handles almost twice that number, or 300 packages per second. It does all this with a fleet of 600+ owned and chartered planes, making it one of the largest airline operators in the world.

When UPS decided it should use linear programming to map its entire operation—every pickup and delivery center and every sorting facility (now nearly 2,000 locations)—to find the best routes to move the millions of packages, it invested close to a decade in developing VOLCANO. This LP-based optimization system (which stands for *Volume, Location, and Aircraft Network Optimization*) is used to determine the least-cost set of routes, fleet assignments, and package flows.

Constraints include the number of planes, airport restrictions, and plane aircraft speed, capacity, and range.

The VOLCANO system is credited with saving UPS hundreds of millions of dollars. But that's just the start. UPS is investing \$600 million more to optimize the whole supply chain to include drivers—the employees closest to the customer—so they will be able to update schedules, priorities, and time conflicts on the fly.

The UPS "airline" is not alone. Southwest runs its massive LP model (called *ILOG Optimizer*) every day to schedule its thousands of flight legs. The program has 90,000 constraints and 2 million variables. United's LP program is called *OptSolver*, and Delta's is called *Coldstart*. Airlines, like many other firms, manage their millions of daily decisions with LP.

Sources: [ups.com](http://ups.com) (June 2015); *Aviation Daily* (February 9, 2004); and *Interfaces* (January–February 2004).

**STUDENT TIP** 

Now we look at three larger problems—ones that have more than two decision variables each and therefore are not graphed.

# Linear Programming Applications

The foregoing examples each contained just two variables ( $X_1$  and  $X_2$ ). Most real-world problems (as we saw in the UPS *OM in Action* box) contain many more variables, however. Let's use the principles already developed to formulate a few more-complex problems. The practice you will get by “paraphrasing” the following LP situations should help develop your skills for applying linear programming to other common operations situations.

**LO B.6** Formulate production-mix, diet, and labor scheduling problems

## Production-Mix Example

Example B2 involves another *production-mix* decision. Limited resources must be allocated among various products that a firm produces. The firm's overall objective is to manufacture the selected products in such quantities as to maximize total profits.

### Example B2

#### A PRODUCTION-MIX PROBLEM

Failsafe Electronics Corporation primarily manufactures four highly technical products, which it supplies to aerospace firms that hold NASA contracts. Each of the products must pass through the following departments before they are shipped: wiring, drilling, assembly, and inspection. The time requirements in each department (in hours) for each unit produced and its corresponding profit value are summarized in this table:

PRODUCT	DEPARTMENT				UNIT PROFIT
	WIRING	DRILLING	ASSEMBLY	INSPECTION	
XJ201	.5	3	2	.5	\$ 9
XM897	1.5	1	4	1.0	\$12
TR29	1.5	2	1	.5	\$15
BR788	1.0	3	2	.5	\$11

The production time available in each department each month and the minimum monthly production requirement to fulfill contracts are as follows:

DEPARTMENT	CAPACITY (HOURS)	PRODUCT	MINIMUM PRODUCTION LEVEL
Wiring	1,500	XJ201	150
Drilling	2,350	XM897	100
Assembly	2,600	TR29	200
Inspection	1,200	BR788	400

**APPROACH** ► Formulate this production-mix situation as an LP problem. The production manager first specifies production levels for each product for the coming month. He lets:

$X_1$  = number of units of XJ201 produced

$X_2$  = number of units of XM897 produced

$X_3$  = number of units of TR29 produced

$X_4$  = number of units of BR788 produced

**SOLUTION** ► The LP formulation is:

Objective: Maximize profit =  $9X_1 + 12X_2 + 15X_3 + 11X_4$

subject to:

$$.5X_1 + 1.5X_2 + 1.5X_3 + 1X_4 \leq 1,500 \text{ hours of wiring available}$$

$$3X_1 + 1X_2 + 2X_3 + 3X_4 \leq 2,350 \text{ hours of drilling available}$$

$$2X_1 + 4X_2 + 1X_3 + 2X_4 \leq 2,600 \text{ hours of assembly available}$$

$$.5X_1 + 1X_2 + .5X_3 + .5X_4 \leq 1,200 \text{ hours of inspection}$$

$X_1 \geq 150$  units of XJ201

$X_2 \geq 100$  units of XM897

$X_3 \geq 200$  units of TR29

$X_4 \geq 400$  units of BR788

$X_1, X_2, X_3, X_4 \geq 0$

**INSIGHT** ▶ There can be numerous constraints in an LP problem. The constraint right-hand sides may be in different units, but the objective function uses one common unit—dollars of profit, in this case. Because there are more than two decision variables, this problem is not solved graphically.

**LEARNING EXERCISE** ▶ Solve this LP problem as formulated. What is the solution? [Answer:  $X_1 = 150$ ,  $X_2 = 300$ ,  $X_3 = 200$ ,  $X_4 = 400$ .]

**RELATED PROBLEMS** ▶ B.5–B.8, B.10–B.14, B.37 (B.15, B.17, B.19, B.21, B.24 are available in MyOMLab)

## Diet Problem Example

Example B3 illustrates the *diet problem*, which was originally used by hospitals to determine the most economical diet for patients. Known in agricultural applications as the *feed-mix problem*, the diet problem involves specifying a food or feed ingredient combination that will satisfy stated nutritional requirements at a minimum cost level.

### Example B3

#### A DIET PROBLEM

The Feed 'N Ship feedlot fattens cattle for local farmers and ships them to meat markets in Kansas City and Omaha. The owners of the feedlot seek to determine the amounts of cattle feed to buy to satisfy minimum nutritional standards and, at the same time, minimize total feed costs.

Each grain stock contains different amounts of four nutritional ingredients: A, B, C, and D. Here are the ingredient contents of each grain, in *ounces per pound of grain*:

INGREDIENT	FEED		
	STOCK X	STOCK Y	STOCK Z
A	3 oz	2 oz	4 oz
B	2 oz	3 oz	1 oz
C	1 oz	0 oz	2 oz
D	6 oz	8 oz	4 oz

The cost per pound of grains X, Y, and Z is \$0.02, \$0.04, and \$0.025, respectively. The minimum requirement per cow per month is 64 ounces of ingredient A, 80 ounces of ingredient B, 16 ounces of ingredient C, and 128 ounces of ingredient D.

The feedlot faces one additional restriction—it can obtain only 500 pounds of stock Z per month from the feed supplier, regardless of its need. Because there are usually 100 cows at the Feed 'N Ship feedlot at any given time, this constraint limits the amount of stock Z for use in the feed of each cow to no more than 5 pounds, or 80 ounces, per month.

**APPROACH** ▶ Formulate this as a minimization LP problem.

Let:  $X_1$  = number of pounds of stock X purchased per cow each month  
 $X_2$  = number of pounds of stock Y purchased per cow each month  
 $X_3$  = number of pounds of stock Z purchased per cow each month

**SOLUTION** ▶

Objective: Minimize cost =  $.02X_1 + .04X_2 + .025X_3$   
 subject to: Ingredient A requirement:  $3X_1 + 2X_2 + 4X_3 \geq 64$   
 Ingredient B requirement:  $2X_1 + 3X_2 + 1X_3 \geq 80$   
 Ingredient C requirement:  $1X_1 + 0X_2 + 2X_3 \geq 16$   
 Ingredient D requirement:  $6X_1 + 8X_2 + 4X_3 \geq 128$   
 Stock Z limitation:  $X_3 \leq 5$   
 $X_1, X_2, X_3 \geq 0$

The cheapest solution is to purchase 40 pounds of grain  $X_1$ , at a cost of \$0.80 per cow.

**INSIGHT** ▶ Because the cost per pound of stock X is so low, the optimal solution excludes grains Y and Z.

**LEARNING EXERCISE** ▶ The cost of a pound of stock X just increased by 50%. Does this affect the solution? [Answer: Yes, when the cost per pound of grain X is \$0.03,  $X_1 = 16$  pounds,  $X_2 = 16$  pounds,  $X_3 = 0$ , and cost = \$1.12 per cow.]

**RELATED PROBLEMS** ▶ B.27, B.28, B.40 (B.33 is available in MyOMLab)

## Labor Scheduling Example

*Labor scheduling problems* address staffing needs over a specific time period. They are especially useful when managers have some flexibility in assigning workers to jobs that require overlapping or interchangeable talents. Large banks and hospitals frequently use LP to tackle their labor scheduling. Example B4 describes how one bank uses LP to schedule tellers.

### Example B4

#### SCHEDULING BANK TELLERS

Mexico City Bank of Commerce and Industry is a busy bank that has requirements for between 10 and 18 tellers depending on the time of day. Lunchtime, from noon to 2 P.M., is usually heaviest. The table below indicates the workers needed at various hours that the bank is open.

TIME PERIOD	NUMBER OF TELLERS REQUIRED	TIME PERIOD	NUMBER OF TELLERS REQUIRED
9 A.M.–10 A.M.	10	1 P.M.–2 P.M.	18
10 A.M.–11 A.M.	12	2 P.M.–3 P.M.	17
11 A.M.–Noon	14	3 P.M.–4 P.M.	15
Noon–1 P.M.	16	4 P.M.–5 P.M.	10

The bank now employs 12 full-time tellers, but many people are on its roster of available part-time employees. A part-time employee must put in exactly 4 hours per day but can start anytime between 9 A.M. and 1 P.M. Part-timers are a fairly inexpensive labor pool because no retirement or lunch benefits are provided them. Full-timers, on the other hand, work from 9 A.M. to 5 P.M. but are allowed 1 hour for lunch. (Half the full-timers eat at 11 A.M., the other half at noon.) Full-timers thus provide 35 hours per week of productive labor time.

By corporate policy, the bank limits part-time hours to a maximum of 50% of the day's total requirement. Part-timers earn \$6 per hour (or \$24 per day) on average, whereas full-timers earn \$75 per day in salary and benefits on average.

**APPROACH** ► The bank would like to set a schedule, using LP, that would minimize its total manpower costs. It will release 1 or more of its full-time tellers if it is profitable to do so.

We can let:

- $F$  = full-time tellers
- $P_1$  = part-timers starting at 9 A.M. (leaving at 1 P.M.)
- $P_2$  = part-timers starting at 10 A.M. (leaving at 2 P.M.)
- $P_3$  = part-timers starting at 11 A.M. (leaving at 3 P.M.)
- $P_4$  = part-timers starting at noon (leaving at 4 P.M.)
- $P_5$  = part-timers starting at 1 P.M. (leaving at 5 P.M.)

**SOLUTION** ► Objective function:

$$\text{Minimize total daily manpower cost} = \$75F + \$24(P_1 + P_2 + P_3 + P_4 + P_5)$$

Constraints: For each hour, the available labor-hours must be at least equal to the required labor-hours:

$$\begin{aligned} F + P_1 &\geq 10 && (9 \text{ A.M. to } 10 \text{ A.M. needs}) \\ F + P_1 + P_2 &\geq 12 && (10 \text{ A.M. to } 11 \text{ A.M. needs}) \\ \frac{1}{2}F + P_1 + P_2 + P_3 &\geq 14 && (11 \text{ A.M. to noon needs}) \\ \frac{1}{2}F + P_1 + P_2 + P_3 + P_4 &\geq 16 && (\text{noon to } 1 \text{ P.M. needs}) \\ F + P_2 + P_3 + P_4 + P_5 &\geq 18 && (1 \text{ P.M. to } 2 \text{ P.M. needs}) \\ F + P_3 + P_4 + P_5 &\geq 17 && (2 \text{ P.M. to } 3 \text{ P.M. needs}) \\ F + P_4 + P_5 &\geq 15 && (3 \text{ P.M. to } 4 \text{ P.M. needs}) \\ F + P_5 &\geq 10 && (4 \text{ P.M. to } 5 \text{ P.M. needs}) \end{aligned}$$

Only 12 full-time tellers are available, so:

$$F \leq 12$$

Part-time worker-hours cannot exceed 50% of total hours required each day, which is the sum of the tellers needed each hour:

$$4(P_1 + P_2 + P_3 + P_4 + P_5) \leq .50(10 + 12 + 14 + 16 + 18 + 17 + 15 + 10)$$

or:

$$4P_1 + 4P_2 + 4P_3 + 4P_4 + 4P_5 \leq 0.50(112)$$

$$F, P_1, P_2, P_3, P_4, P_5 \geq 0$$

There are two alternative optimal schedules that Mexico City Bank can follow. The first is to employ only 10 full-time tellers ( $F = 10$ ) and to start 7 part-timers at 10 A.M. ( $P_2 = 7$ ), 2 part-timers at 11 A.M. and noon ( $P_3 = 2$  and  $P_4 = 2$ ), and 3 part-timers at 1 P.M. ( $P_5 = 3$ ). No part-timers would begin at 9 A.M.

The second solution also employs 10 full-time tellers, but starts 6 part-timers at 9 A.M. ( $P_1 = 6$ ), 1 part-timer at 10 A.M. ( $P_2 = 1$ ), 2 part-timers at 11 A.M. and noon ( $P_3 = 2$  and  $P_4 = 2$ ), and 3 part-timers at 1 P.M. ( $P_5 = 3$ ). The cost of either of these two policies is \$1,086 per day.

**INSIGHT** ► It is not unusual for multiple optimal solutions to exist in large LP problems. In this case, it gives management the option of selecting, at the same cost, between schedules. To find an alternate optimal solution, you may have to enter the constraints in a different sequence.

**LEARNING EXERCISE** ► The bank decides to give part-time employees a raise to \$7 per hour. Does the solution change? [Answer: Yes, cost = \$1,142,  $F = 10$ ,  $P_1 = 6$ ,  $P_2 = 1$ ,  $P_3 = 2$ ,  $P_4 = 5$ ,  $P_5 = 0$ .]

**RELATED PROBLEMS** ► B.36

## The Simplex Method of LP

Most real-world linear programming problems have more than two variables and thus are too complex for graphical solution. A procedure called the **simplex method** may be used to find the optimal solution to such problems. The simplex method is actually an algorithm (or a set of instructions) with which we examine corner points in a methodical fashion until we arrive at the best solution—highest profit or lowest cost. Computer programs (such as Excel OM and POM for Windows) and Excel spreadsheets are available to solve linear programming problems via the simplex method.

For details regarding the algebraic steps of the simplex algorithm, see Tutorial 3 at our text student download site or in MyOMLab, or refer to a management science textbook.<sup>2</sup>

## Integer and Binary Variables

All the examples we have seen in this module so far have produced integer solutions. But it is very common to see LP solutions where the decision variables are not whole numbers. Computer software provides a simple way to guarantee only integer solutions. In addition, computers allow us to create special decision variables called **binary variables** that can only take on the values of 0 or 1. Binary variables allow us to introduce “yes-or-no” decisions into our linear programs and to introduce special logical conditions.

### Creating Integer and Binary Variables

If we wish to ensure that decision variable values are integers rather than fractions, it is generally *not* good practice to simply round the solutions to the nearest integer values. The rounded solutions may not be optimal and, in fact, may not even be feasible. Fortunately, all LP software programs have simple ways to add constraints that enforce some or all of the decision variables to be either integer or binary. The main disadvantage of introducing such constraints is that larger programs may take longer to solve. The same LP that may take 3 seconds to solve on a computer could take several hours or more to solve if many of its variables are forced to be integer or binary. For relatively small programs, though, the difference may be unnoticeable.

Using Excel’s Solver (see *Using Software to Solve LP Problems* later in this module), integer and binary constraints can be added by clicking **Add** from the main Solver dialog box. Using the **Add Constraint** dialog box (see Program B.2), highlight the decision variables themselves under **Cell Reference**:. Then select **int** or **bin** to ensure that those variables are integer or binary, respectively, in the optimal solution.

### Simplex method

An algorithm for solving linear programming problems of all sizes.

### Binary variables

Decision variables that can only take on the value of 0 or 1.

## Program B.2

Excel's Solver Dialog Box to Add Integer or Binary Constraints on Variables



## Linear Programming Applications with Binary Variables

In the written formulation of a linear program, binary variables are usually defined using the following form:

$$Y = \begin{cases} 1 & \text{if some condition holds} \\ 0 & \text{otherwise} \end{cases}$$

Sometimes we designate decision variables as binary if we are making a yes-or-no decision; for example, “Should we undertake this particular project?” “Should we buy that machine?” or “Should we locate a facility in Arkansas?” Other times, we create binary variables to introduce additional logic into our programs.

**Limiting the Number of Alternatives Selected** One common use of 0-1 variables involves limiting the number of projects or items that are selected from a group. Suppose a firm is required to select no more than two of three potential projects. This could be modeled with the following constraint:

$$Y_1 + Y_2 + Y_3 \leq 2$$

If we wished to force the selection of *exactly* two of the three projects for funding, the following constraint should be used:

$$Y_1 + Y_2 + Y_3 = 2$$

This forces exactly two of the variables to have values of 1, whereas the other variable must have a value of 0.

**Dependent Selections** At times the selection of one project depends in some way on the selection of another project. This situation can be modeled with the use of 0-1 variables. Suppose G.E.’s new catalytic converter could be purchased ( $Y_1 = 1$ ) only if the software was also purchased ( $Y_2 = 1$ ). The following constraint would force this to occur:

$$Y_1 \leq Y_2$$

or, equivalently,

$$Y_1 - Y_2 \leq 0$$

Thus, if the software is not purchased, the value of  $Y_2$  is 0, and the value of  $Y_1$  must also be 0 because of this constraint. However, if the software is purchased ( $Y_2 = 1$ ), then it is possible that the catalytic converter could also be purchased ( $Y_1 = 1$ ), although this is not required.

If we wished for the catalytic converter and the software projects to either both be selected or both not be selected, we should use the following constraint:

$$Y_1 = Y_2$$

or, equivalently,

$$Y_1 - Y_2 = 0$$

Thus, if either of these variables is equal to 0, the other must also be 0. If either of these is equal to 1, the other must also be 1.



## A Fixed-Charge Integer Programming Problem

Often businesses are faced with decisions involving a fixed charge that will affect the cost of future operations. Building a new factory or entering into a long-term lease on an existing facility would involve a fixed cost that might vary depending on the size of the facility and the location. Once a factory is built, the variable production costs will be affected by the labor cost in the particular city where it is located. Example B5 provides an illustration.

### Example B5

#### A FIXED-CHARGE PROBLEM USING BINARY VARIABLES

Sitka Manufacturing is planning to build at least one new plant, and three cities are being considered: Baytown, Texas; Lake Charles, Louisiana; and Mobile, Alabama. Once the plant or plants have been constructed, the company wishes to have sufficient capacity to produce at least 38,000 units each year. The costs associated with the possible locations are given in the following table.

SITE	ANNUAL FIXED COST	VARIABLE COST PER UNIT	ANNUAL CAPACITY
Baytown, TX	\$340,000	\$32	21,000
Lake Charles, LA	\$270,000	\$33	20,000
Mobile, AL	\$290,000	\$30	19,000

**APPROACH** ► In modeling this as an integer program, the objective function is to minimize the total of the fixed costs and the variable costs. The constraints are: (1) total production capacity is at least 38,000; (2) the number of units produced at the Baytown plant is 0 if the plant is not built, and it is no more than 21,000 if the plant is built; (3) the number of units produced at the Lake Charles plant is 0 if the plant is not built, and it is no more than 20,000 if the plant is built; and (4) the number of units produced at the Mobile plant is 0 if the plant is not built, and it is no more than 19,000 if the plant is built.

Then, we define the decision variables as

$$Y_1 = \begin{cases} 1 & \text{if factory is built in Baytown} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_2 = \begin{cases} 1 & \text{if factory is built in Lake Charles} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_3 = \begin{cases} 1 & \text{if factory is built in Mobile} \\ 0 & \text{otherwise} \end{cases}$$

$X_1$  = number of units produced at the Baytown plant

$X_2$  = number of units produced at the Lake Charles plant

$X_3$  = number of units produced at the Mobile plant

**SOLUTION** ► The integer programming problem formulation becomes

Objective: Minimize cost =  $340,000 Y_1 + 270,000 Y_2 + 290,000 Y_3 + 32X_1 + 33X_2 + 30X_3$

subject to:  $X_1 + X_2 + X_3 \geq 38,000$

$$X_1 \leq 21,000 Y_1$$

$$X_2 \leq 20,000 Y_2$$

$$X_3 \leq 19,000 Y_3$$

$$X_1, X_2, X_3 \geq 0 \text{ and integer}$$

$$Y_1, Y_2, Y_3 = 0 \text{ or } 1$$

**INSIGHT** ► Examining the second constraint, the objective function will try to set the binary variable  $Y_1$  equal to 0 because it wants to minimize cost. However, if  $Y_1 = 0$ , then the constraint will force  $X_1$  to equal 0, in which case no units will be produced, and the plant will not be opened. Alternatively, if the rest of the program deems it worthwhile or necessary to produce some units of  $X_1$ , then  $Y_1$  will have to equal 1 for the constraint to hold. And when  $Y_1 = 1$ , the firm will be charged the fixed cost of \$340,000, and production will be limited to the capacity of 21,000 units. The same logic applies for constraints 3 and 4.

**LEARNING EXERCISE** ► Solve this integer program as formulated. What is the solution? [Answer:  $Y_1 = 0, Y_2 = 1, Y_3 = 1, X_1 = 0, X_2 = 19,000, X_3 = 19,000$ ; Total Cost = \$1,757,000.]

**RELATED PROBLEMS** ► B.41, B.42

## Summary

This module introduces a special kind of model, linear programming. LP has proven to be especially useful when trying to make the most effective use of an organization's resources.

The first step in dealing with LP models is problem formulation, which involves identifying and creating an objective function and constraints. The second step is to solve

the problem. If there are only two decision variables, the problem can be solved graphically, using the corner-point method or the iso-profit/iso-cost line method. With either approach, we first identify the feasible region, then find the corner point yielding the greatest profit or least cost. LP is used in a wide variety of business applications, as the examples and homework problems in this module reveal.

### Key Terms

Linear programming (LP) (p. 700)

Objective function (p. 701)

Constraints (p. 701)

Graphical solution approach (p. 702)

Decision variables (p. 702)

Feasible region (p. 703)

Iso-profit line method (p. 703)

Corner-point method (p. 705)

Parameter (p. 705)

Sensitivity analysis (p. 706)

Shadow price (or dual value) (p. 707)

Iso-cost (p. 708)

Simplex method (p. 713)

Binary variables (p. 713)

### Discussion Questions

- List at least four applications of linear programming problems.
- What is a "corner point"? Explain why solutions to linear programming problems focus on corner points.
- Define the feasible region of a graphical LP problem. What is a feasible solution?
- Each linear programming problem that has a feasible region has an infinite number of solutions. Explain.
- Under what circumstances is the objective function more important than the constraints in a linear programming model?
- Under what circumstances are the constraints more important than the objective function in a linear programming model?
- Why is the diet problem, in practice, applicable for animals but not particularly for people?
- How many feasible solutions are there in a linear program? Which ones do we need to examine to find the optimal solution?
- Define shadow price (or dual value).
- Explain how to use the iso-cost line in a graphical minimization problem.
- Compare how the corner-point and iso-profit line methods work for solving graphical problems.
- Where a constraint crosses the vertical or horizontal axis, the quantity is fairly obvious. How does one go about finding the quantity coordinates where two constraints cross, not at an axis?
- Suppose a linear programming (maximization) problem has been solved and that the optimal value of the objective function is \$300. Suppose an additional constraint is added to this problem. Explain how this might affect each of the following:
  - The feasible region.
  - The optimal value of the objective function.

### Using Software to Solve LP Problems

All LP problems can be solved with the simplex method, using software such as Excel, Excel OM, or POM for Windows.

#### ✕ CREATING YOUR OWN EXCEL SPREADSHEETS

Excel offers the ability to analyze linear programming problems using built-in problem-solving tools. Excel's tool is named *Solver*.

We use Excel to set up the Glickman Electronics problem in Program B.3. The objective and constraints are repeated here:

Objective function: Maximize profit = \$7(No. of x-pods) + \$5(No. of BlueBerrys)

Subject to:  $4(x\text{-pods}) + 3(\text{BlueBerrys}) \leq 240$

$2(x\text{-pods}) + 1(\text{BlueBerry}) \leq 100$

#### Program B.3

##### Using Excel to Formulate the Glickman Electronics Problem

The screenshot shows an Excel spreadsheet with the following data:

	x-pods	BlueBerrys		
1	Glickman Electronics			
2	Using Linear Programming to Solve a Product Mix Problem			
3				
4				
5	Variable Values	0	0	
6	Objective Function	\$7.00	\$5.00	\$0.00
7	Subject to:			
8	Electronic Time Available	4	3	0 <= 240
9	Assembly Time Available	2	1	0 <= 100
10				

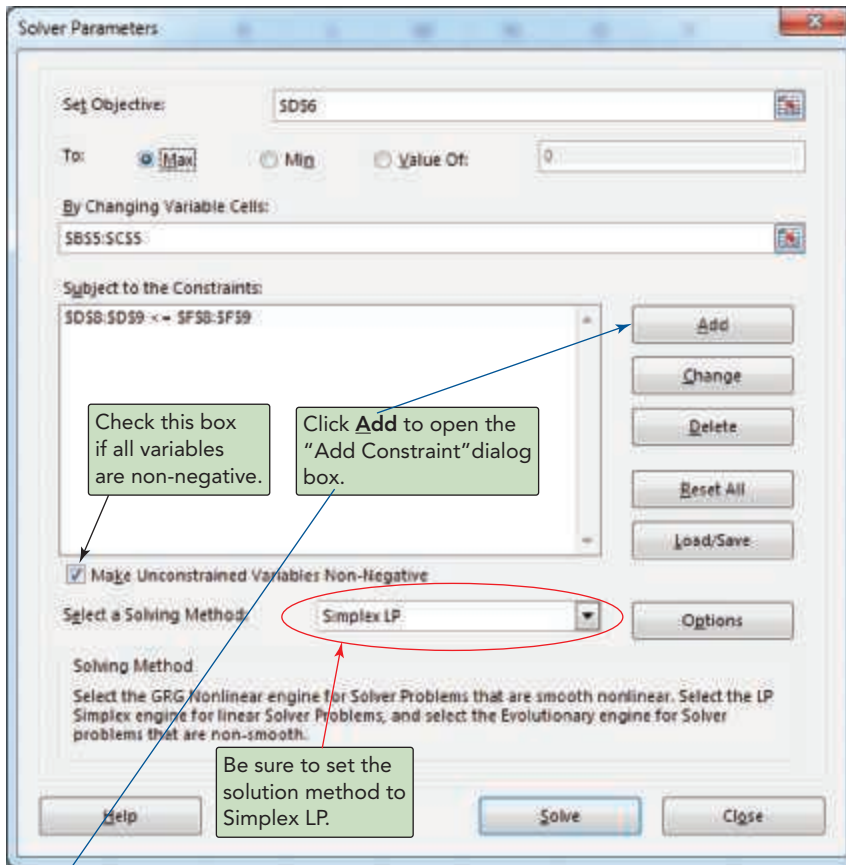
Annotations in the image:

- "The decisions (the number of units to produce) go here." points to the cells containing 0 and 0 in row 5.
- "These are simply labels." points to the column headers 'x-pods' and 'BlueBerrys' in row 4.
- "Action Copy D6 to D8:D9" points to the cell containing \$0.00 in row 6, column D.
- "The objective function value (profit) goes here." points to the cell containing \$0.00 in row 6, column D.
- "=B6\*\$B\$5+C6\*\$C\$5" is shown in the formula bar.

To ensure that Solver always loads when Excel is loaded, click on **FILE**, then **Options**, then **Add-Ins**. Next to **Manage**: at the bottom, make sure that **Excel Add-Ins** is selected, and click on the **Go...** button. Check **Solver Add-In**, and click **OK**. Once in Excel, the Solver dialog box will appear by clicking on: **Data**, then **Analysis: Solver**. (Or if using Excel for Mac, select **Tools, Solver**.) Program B.4 shows how to use Solver to find the optimal (very best) solution to the Glickman Electronics problem. Click on **Solve**, and the solution will automatically appear in the spreadsheet in the green and blue cells.

Program **B.4**

Solver Dialog Boxes for the Glickman Electronics Problem



The Excel screen in Program B.5 shows Solver’s solution to the Glickman Electronics Company problem. Note that the optimal solution is now shown in cells B5 and C5, which serve as the variables. The Reports selections perform more extensive analysis of the solution and its environment. Excel’s sensitivity analysis capability was illustrated earlier in Program B.1.

Program **B.5**

Excel Solution to Glickman Electronics LP Problem

	A	B	C	D	E	F
1	Glickman Electronics					
2	Using Linear Programming to Solve a Product Mix Problem					
3						
4		x-pods	BlueBerrys			
5	Variable Values	30	40			
6	Objective Function	\$7.00	\$5.00	\$410.00	Max	RHS
7	Subject to:					
8	Electronic Time Available	4	3	240	<=	240
9	Assembly Time Available	2	1	100	<=	100

**PX USING EXCEL OM AND POM FOR WINDOWS**

Excel OM and POM for Windows can handle relatively large LP problems. As output, the software provides optimal values for the variables, optimal profit or cost, and sensitivity analysis. In addition, POM for Windows provides graphical output for problems with only two variables.

## Solved Problems

Virtual Office Hours help is available in [MyOMLab](#).

**SOLVED PROBLEM B.1**

Smith's, a Niagara, New York, clothing manufacturer that produces men's shirts and pajamas, has two primary resources available: sewing-machine time (in the sewing department) and cutting-machine time (in the cutting department). Over the next month, owner Barbara Smith can schedule up to 280 hours of work on sewing machines and up to 450 hours of work on cutting machines. Each shirt produced requires

1.00 hour of sewing time and 1.50 hours of cutting time. Producing each pair of pajamas requires .75 hours of sewing time and 2 hours of cutting time.

To express the LP constraints for this problem mathematically, we let:

$X_1$  = number of shirts produced

$X_2$  = number of pajamas produced

**SOLUTION**

First constraint:  $1X_1 + .75X_2 \leq 280$  hours of sewing-machine time available—our first scarce resource

Second constraint:  $1.5X_1 + 2X_2 \leq 450$  hours of cutting-machine time available—our second scarce resource

*Note:* This means that each pair of pajamas takes 2 hours of the cutting resource. Smith's accounting department analyzes cost and sales figures and states that each shirt produced will yield a \$4 contribution to profit and that each pair of pajamas will yield a \$3 contribution to profit.

This information can be used to create the LP *objective function* for this problem:

Objective function: Maximize total contribution to profit =  $\$4X_1 + \$3X_2$

**SOLVED PROBLEM B.2**

We want to solve the following LP problem for Kevin Caskey Wholesale Inc. using the corner-point method:

Objective: Maximize profit =  $\$9X_1 + \$7X_2$

Constraints:  $2X_1 + 1X_2 \leq 40$   
 $X_1 + 3X_2 \leq 30$   
 $X_1, X_2 \geq 0$

**SOLUTION**

Figure B.10 illustrates these constraints:

Corner-point a: ( $X_1 = 0, X_2 = 0$ ) Profit = 0

Corner-point b: ( $X_1 = 0, X_2 = 10$ ) Profit =  $9(0) + 7(10) = \$70$

Corner-point d: ( $X_1 = 20, X_2 = 0$ ) Profit =  $9(20) + 7(0) = \$180$

Corner-point c is obtained by solving equations  $2X_1 + 1X_2 = 40$  and  $X_1 + 3X_2 = 30$  simultaneously. Multiply the second equation by  $-2$  and add it to the first.

$$\begin{array}{r} 2X_1 + 1X_2 = 40 \\ -2X_1 - 6X_2 = -60 \\ \hline -5X_2 = -20 \\ \text{Thus } X_2 = 4 \end{array}$$

And  $X_1 + 3(4) = 30$  or  $X_1 + 12 = 30$  or  $X_1 = 18$

Corner-point c: ( $X_1 = 18, X_2 = 4$ ) Profit =  $9(18) + 7(4) = \$190$

Hence the optimal solution is:

$$(x_1 = 18, x_2 = 4) \quad \text{Profit} = \$190$$

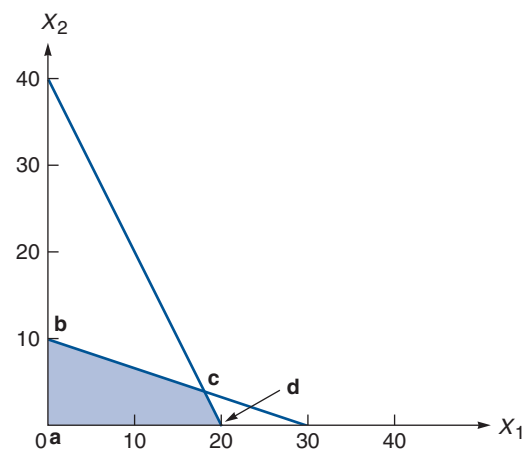


Figure B.10

K. Caskey Wholesale Inc.'s Feasible Region

**SOLVED PROBLEM B.3**

Holiday Meal Turkey Ranch is considering buying two different types of turkey feed. Each feed contains, in varying proportions, some or all of the three nutritional ingredients essential for fattening turkeys. Brand Y feed costs the ranch \$.02 per pound. Brand Z costs \$.03 per pound. The rancher would like to determine the lowest-cost diet that meets the minimum monthly intake requirement for each nutritional ingredient.

The following table contains relevant information about the composition of brand Y and brand Z feeds, as well as the minimum monthly requirement for each nutritional ingredient per turkey.

COMPOSITION OF EACH POUND OF FEED			
INGREDIENT	BRAND Y FEED	BRAND Z FEED	MINIMUM MONTHLY REQUIREMENT
A	5 oz	10 oz	90 oz
B	4 oz	3 oz	48 oz
C	.5 oz	0	1.5 oz
Cost/lb	\$.02	\$.03	

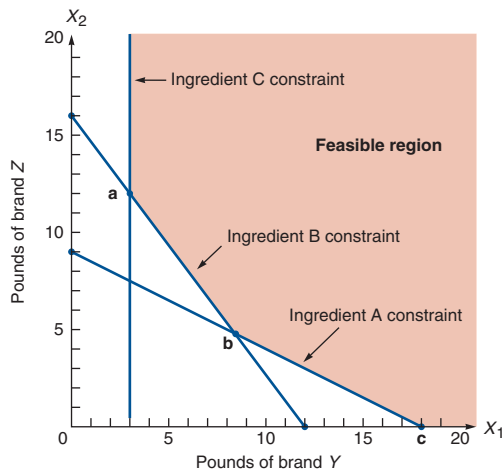


Figure B.11

**Feasible Region for the Holiday Meal Turkey Ranch Problem**

**STUDENT TIP**

Note that the last line parallel to the 54¢ iso-cost line that touches the feasible region indicates the optimal corner point.

**SOLUTION**

If we let:

$X_1$  = number of pounds of brand Y feed purchased  
 $X_2$  = number of pounds of brand Z feed purchased

then we may proceed to formulate this linear programming problem as follows:

Objective: Minimize cost (in cents) =  $2X_1 + 3X_2$

subject to these constraints:

$5X_1 + 10X_2 \geq 90$  oz (ingredient A constraint)  
 $4X_1 + 3X_2 \geq 48$  oz (ingredient B constraint)  
 $\frac{1}{2}X_1 \geq 1\frac{1}{2}$  oz (ingredient C constraint)

Figure B.11 illustrates these constraints.

The iso-cost line approach may be used to solve LP minimization problems such as that of the Holiday Meal Turkey Ranch. As with iso-profit lines, we need not compute the cost at each corner point, but instead draw a series of parallel cost lines. The last cost point to touch the feasible region provides us with the optimal solution corner.

For example, we start in Figure B.12 by drawing a 54¢ cost line, namely,  $54 = 2X_1 + 3X_2$ . Obviously, there are many points in the feasible region that would yield a lower total cost. We proceed to move our iso-cost line toward the lower left, in a plane parallel to the 54¢ solution line. The last point we touch while still in contact with the feasible region is the same as corner point **b** of Figure B.11. It has the coordinates ( $X_1 = 8.4$ ,  $X_2 = 4.8$ ) and an associated cost of 31.2 cents.

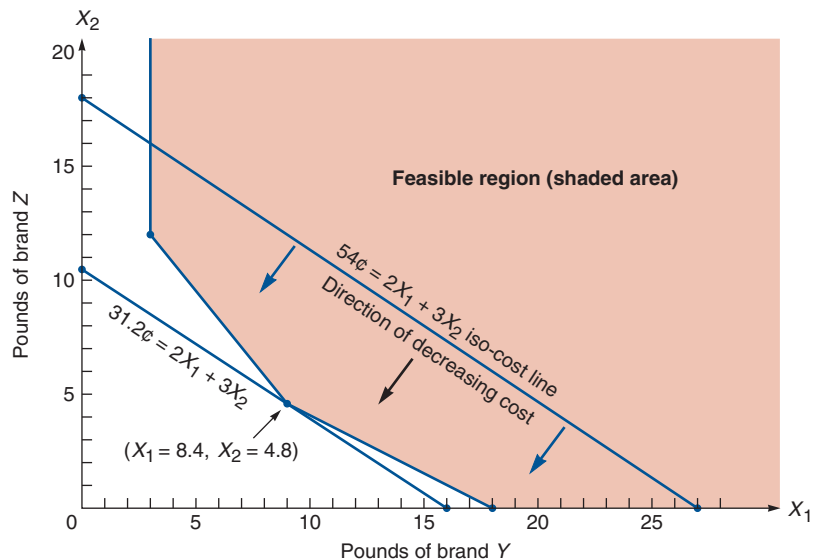


Figure B.12

**Graphical Solution to the Holiday Meal Turkey Ranch Problem Using the Iso-Cost Line**

## Problems

Note: **Px** means the problem may be solved with POM for Windows and/or Excel OM.

**Problem B.1 relates to Requirements of a Linear Programming Problem**

- **B.1** The LP relationships that follow were formulated by Richard Martin at the Long Beach Chemical Company. Which ones are invalid for use in a linear programming problem, and why?

$$\begin{aligned} \text{Maximize} &= 6X_1 + \frac{1}{2}X_1X_2 + 5X_3 \\ \text{Subject to:} & \quad 4X_1X_2 + 2X_3 \leq 70 \\ & \quad 7.9X_1 - 4X_2 \geq 15.6 \\ & \quad 3X_1 + 3X_2 + 3X_3 \geq 21 \\ & \quad 19X_2 - \frac{1}{3}X_3 = 17 \\ & \quad -X_1 - X_2 + 4X_3 = 5 \\ & \quad 4X_1 + 2X_2 + 3\sqrt{X_3} \leq 80 \end{aligned}$$

**Problems B.2–B.21 relate to Graphical Solution to a Linear Programming Problem**

- **B.2** Solve the following linear programming problem graphically:

$$\begin{aligned} \text{Maximize profit} &= 4X + 6Y \\ \text{Subject to:} & \quad X + 2Y \leq 8 \\ & \quad 5X + 4Y \leq 20 \\ & \quad X, Y \geq 0 \end{aligned} \quad \mathbf{Px}$$

- **B.3** Solve the following linear programming problem graphically:

$$\begin{aligned} \text{Maximize profit} &= X + 10Y \\ \text{Subject to:} & \quad 4X + 3Y \leq 36 \\ & \quad 2X + 4Y \leq 40 \\ & \quad Y \geq 3 \\ & \quad X, Y \geq 0 \end{aligned} \quad \mathbf{Px}$$

- **B.4** Consider the following linear programming problem:

$$\begin{aligned} \text{Maximize profit} &= 30X_1 + 10X_2 \\ \text{Subject to:} & \quad 3X_1 + X_2 \leq 300 \\ & \quad X_1 + X_2 \leq 200 \\ & \quad X_1 \leq 100 \\ & \quad X_2 \geq 50 \\ & \quad X_1 - X_2 \leq 0 \\ & \quad X_1, X_2 \geq 0 \end{aligned}$$

- Solve the problem graphically.
- Is there more than one optimal solution? Explain. **Px**

- **B.5** The Attaran Corporation manufactures two electrical products: portable air conditioners and portable heaters. The assembly process for each is similar in that both require a certain amount of wiring and drilling. Each air conditioner takes 3 hours of wiring and 2 hours of drilling. Each heater must go through 2 hours of wiring and 1 hour of drilling. During the next production period, 240 hours of wiring time are available and up to 140 hours of drilling time may be used. Each air conditioner sold yields a profit of \$25. Each heater assembled may be sold for a \$15 profit.

Formulate and solve this LP production-mix situation, and find the best combination of air conditioners and heaters that yields the highest profit. **Px**

- **B.6** The Chris Beehner Company manufactures two lines of designer yard gates, called model A and model B. Every gate requires blending a certain amount of steel and zinc; the company

has available a total of 25,000 lb of steel and 6,000 lb of zinc. Each model A gate requires a mixture of 125 lb of steel and 20 lb of zinc, and each yields a profit of \$90. Each model B gate requires 100 lb of steel and 30 lb of zinc and can be sold for a profit of \$70.

Find by graphical linear programming the best production mix of yard gates. **Px**

- **B.7** Green Vehicle Inc. manufactures electric cars and small delivery trucks. It has just opened a new factory where the C1 car and the T1 truck can both be manufactured. To make either vehicle, processing in the assembly shop and in the paint shop are required. It takes 1/40 of a day and 1/60 of a day to paint a truck of type T1 and a car of type C1 in the paint shop, respectively. It takes 1/50 of a day to assemble either type of vehicle in the assembly shop.

A T1 truck and a C1 car yield profits of \$300 and \$220, respectively, per vehicle sold.

- Define the objective function and constraint equations.
- Graph the feasible region.
- What is a maximum-profit daily production plan at the new factory?
- How much profit will such a plan yield, assuming whatever is produced is sold? **Px**

- **B.8** The Lifang Wu Corporation manufactures two models of industrial robots, the Alpha 1 and the Beta 2. The firm employs 5 technicians, working 160 hours each per month, on its assembly line. Management insists that full employment (that is, *all* 160 hours of time) be maintained for each worker during next month's operations. It requires 20 labor-hours to assemble each Alpha 1 robot and 25 labor-hours to assemble each Beta 2 model. Wu wants to see at least 10 Alpha 1s and at least 15 Beta 2s produced during the production period. Alpha 1s generate a \$1,200 profit per unit, and Beta 2s yield \$1,800 each.

Determine the most profitable number of each model of robot to produce during the coming month. **Px**

- **B.9** Consider the following LP problem developed at Zafar Malik's Carbondale, Illinois, optical scanning firm:

$$\begin{aligned} \text{Maximize profit} &= \$1X_1 + \$1X_2 \\ \text{Subject to:} & \quad 2X_1 + 1X_2 \leq 100 \\ & \quad 1X_1 + 2X_2 \leq 100 \end{aligned}$$

- What is the optimal solution to this problem? Solve it graphically.
- If a technical breakthrough occurred that raised the profit per unit of  $X_1$  to \$3, would this affect the optimal solution?
- Instead of an increase in the profit coefficient  $X_1$  to \$3, suppose that profit was overestimated and should only have been \$1.25. Does this change the optimal solution? **Px**

- **B.10** A craftsman named William Barnes builds two kinds of birdhouses, one for wrens and a second for bluebirds. Each wren birdhouse takes 4 hours of labor and 4 units of lumber. Each bluebird house requires 2 hours of labor and 12 units of lumber. The craftsman has available 60 hours of labor and 120 units of lumber. Wren houses yield a profit of \$6 each, and bluebird houses yield a profit of \$15 each.

- Write out the objective and constraints.
- Solve graphically. **Px**

•• **B.11** Each coffee table produced by Kevin Watson Designers nets the firm a profit of \$9. Each bookcase yields a \$12 profit. Watson’s firm is small and its resources limited. During any given production period (of 1 week), 10 gallons of varnish and 12 lengths of high-quality redwood are available. Each coffee table requires approximately 1 gallon of varnish and 1 length of redwood. Each bookcase takes 1 gallon of varnish and 2 lengths of wood.

Formulate Watson’s production-mix decision as a linear programming problem, and solve. How many tables and bookcases should be produced each week? What will the maximum profit be? **Px**

• **B.12** Par, Inc., produces a standard golf bag and a deluxe golf bag on a weekly basis. Each golf bag requires time for cutting and dyeing and time for sewing and finishing, as shown in the following table:

HOURS REQUIRED PER BAG		
PRODUCT	CUTTING AND DYEING	SEWING AND FINISHING
Standard bag	1/2	1
Deluxe bag	1	2/3

The profits per bag and weekly hours available for cutting and dyeing and for sewing and finishing are as follows:

PRODUCT	PROFIT PER UNIT (\$)
Standard bag	10
Deluxe bag	8

ACTIVITY	WEEKLY HOURS AVAILABLE
Cutting and dyeing	300
Sewing and finishing	360

Par, Inc., will sell whatever quantities it produces of these two products.

- a) Find the mix of standard and deluxe golf bags to produce per week that maximizes weekly profit from these activities.
- b) What is the value of the profit? **Px**

•• **B.13** The Denver advertising agency promoting the new Breem dishwashing detergent wants to get the best exposure possible for the product within the \$100,000 advertising budget ceiling placed on it. To do so, the agency needs to decide how much of the budget to spend on each of its two most effective media: (1) television spots during the afternoon hours and (2) large ads in the city’s Sunday newspaper. Each television spot costs \$3,000; each Sunday newspaper ad costs \$1,250. The expected exposure, based on industry ratings, is 35,000 viewers for each TV commercial and 20,000 readers for each newspaper advertisement. The agency director, Deborah Kellogg, knows from experience that it is important to use both media in order to reach the broadest spectrum of potential Breem customers. She decides that at least 5 but no more than 25 television spots should be ordered, and that at least 10 newspaper ads should be contracted. How many times should each of the two media be used to obtain maximum exposure while staying within the budget? Use the graphical method to solve. **Px**

•••• **B.14** Baton Rouge’s Mt. Cedar Hospital is a large, private, 600-bed facility complete with laboratories, operating rooms, and X-ray equipment. In seeking to increase revenues, Mt. Cedar’s

administration has decided to make a 90-bed addition on a portion of adjacent land currently used for staff parking. The administrators feel that the labs, operating rooms, and X-ray department are not being fully utilized at present and do not need to be expanded to handle additional patients. The addition of 90 beds, however, involves deciding how many beds should be allocated to the medical staff (for medical patients) and how many to the surgical staff (for surgical patients).

The hospital’s accounting and medical records departments have provided the following pertinent information: The average hospital stay for a medical patient is 8 days, and the average medical patient generates \$2,280 in revenues. The average surgical patient is in the hospital 5 days and generates \$1,515 in revenues. The laboratory is capable of handling 15,000 tests per year more than it was handling. The average medical patient requires 3.1 lab tests, the average surgical patient 2.6 lab tests. Furthermore, the average medical patient uses 1 X-ray, the average surgical patient 2 X-rays. If the hospital were expanded by 90 beds, the X-ray department could handle up to 7,000 X-rays without significant additional cost. Finally, the administration estimates that up to 2,800 additional operations could be performed in existing operating-room facilities. Medical patients, of course, require no surgery, whereas each surgical patient generally has one surgery performed.

Formulate this problem so as to determine how many medical beds and how many surgical beds should be added to maximize revenues. Assume that the hospital is open 365 days per year. **Px**

Additional problems **B.15–B.21** are available in MyOMLab.

**Problems B.22–B.24 relate to Sensitivity Analysis**

•• **B.22** Kalyan Singhal Corp. makes three products, and it has three machines available as resources as given in the following LP problem:

$$\begin{aligned} \text{Maximize contribution} &= 4X_1 + 4X_2 + 7X_3 \\ \text{Subject to: } &1X_1 + 7X_2 + 4X_3 \leq 100 \text{ (hours on machine 1)} \\ &2X_1 + 1X_2 + 7X_3 \leq 110 \text{ (hours on machine 2)} \\ &8X_1 + 4X_2 + 1X_3 \leq 100 \text{ (hours on machine 3)} \end{aligned}$$

- a) Determine the optimal solution using LP software.
- b) Is there unused time available on any of the machines with the optimal solution?
- c) What would it be worth to the firm to make an additional hour of time available on the third machine?
- d) How much would the firm’s profit increase if an extra 10 hours of time were made available on the second machine at no extra cost? **Px**

•••• **B.23** A fertilizer manufacturer has to fulfill supply contracts to its two main customers (650 tons to Customer A and 800 tons to Customer B). It can meet this demand by shipping existing inventory from any of its three warehouses. Warehouse 1 (W1) has 400 tons of inventory on hand, Warehouse 2 (W2) has 500 tons, and Warehouse 3 (W3) has 600 tons. The company would like to arrange the shipping for the lowest cost possible, where the per-ton transit costs are as follows:

	W1	W2	W3
Customer A	\$7.50	\$6.25	\$6.50
Customer B	\$6.75	\$7.00	\$8.00

a) Explain what each of the six decision variables (V) is: (*Hint*: Look at the Solver report below.)

V A1: \_\_\_\_\_  
 V A2: \_\_\_\_\_  
 V A3: \_\_\_\_\_  
 V B1: \_\_\_\_\_  
 V B2: \_\_\_\_\_  
 V B3: \_\_\_\_\_

b) Write out the objective function in terms of the variables (V A1, V A2, etc.) and the objective coefficients.

c) Aside from nonnegativity of the variables, what are the five constraints? Write a short description for each constraint, and write out the formula (and circle the type of equality/inequality).

Description	Variables and Coefficients	What Type?	RHS
C1: _____	Formula: _____	(= >   =   = <)	_____
C2: _____	Formula: _____	(= >   =   = <)	_____
C3: _____	Formula: _____	(= >   =   = <)	_____
C4: _____	Formula: _____	(= >   =   = <)	_____
C5: _____	Formula: _____	(= >   =   = <)	_____

After you formulate and enter the linear program for Problem B.23 in Excel, the Solver gives you the following sensitivity report:

**Variable Cells**

CELL	NAME	FINAL VALUE	REDUCED COST	OBJECTIVE COEFFICIENT	ALLOWABLE INCREASE	ALLOWABLE DECREASE
\$B\$6	V A1	0	1.5	7.5	1E+30	1.5
\$C\$6	V A2	100	0	6.25	0.25	0.75
\$D\$6	V A3	550	0	6.5	0.75	0.25
\$E\$6	V B1	400	0	6.75	0.5	1E+30
\$F\$6	V B2	400	0	7	0.75	0.5
\$G\$6	V B3	0	0.75	8	1E+30	0.75

**Constraints**

CELL	NAME	FINAL VALUE	SHADOW PRICE	CONSTRAINT R.H. SIDE	ALLOWABLE INCREASE	ALLOWABLE DECREASE
\$H\$7	C1	650	6.5	650	50	550
\$H\$8	C2	800	7.25	800	50	400
\$H\$9	C3	400	-0.5	400	400	50
\$H\$10	C4	500	-0.25	500	550	50
\$H\$11	C5	550	0	600	1E+30	50

d) How many of the constraints are binding?

e) What is the range of optimality on variable V A3?

f) If we could ship 10 tons less to Customer A, how much money *might* we be able to save? If we could choose to short *either* Customer A or Customer B by 10 tons, which would we prefer to short? Why? **Px**

Additional problem B.24 is available in MyOMLab.

**Problems B.25–B.33 relate to Solving Minimization Problems**

• **B.25** Solve the following linear program graphically:

$$\begin{aligned} \text{Minimize cost} &= X_1 + X_2 \\ 8X_1 + 16X_2 &\geq 64 \\ X_1 &\geq 0 \\ X_2 &\geq -2 \end{aligned}$$

(*Note*:  $X_2$  values can be negative in this problem.)

• **B.26** Solve the following LP problem graphically:

$$\begin{aligned} \text{Minimize cost} &= 24X + 15Y \\ \text{Subject to:} & \quad 7X + 11Y \geq 77 \\ & \quad 16X + 4Y \geq 80 \\ & \quad X, Y \geq 0 \end{aligned}$$

•• **B.27** Doug Turner Food Processors wishes to introduce a new brand of dog biscuits composed of chicken- and liver-flavored biscuits that meet certain nutritional requirements. The liver-flavored biscuits contain 1 unit of nutrient A and 2 units of nutrient B; the chicken-flavored biscuits contain 1 unit of nutrient A and 4 units of nutrient B. According to federal requirements, there must be at



least 40 units of nutrient A and 60 units of nutrient B in a package of the new mix. In addition, the company has decided that there can be no more than 15 liver-flavored biscuits in a package. If it costs 1¢ to make 1 liver-flavored biscuit and 2¢ to make 1 chicken-flavored, what is the optimal product mix for a package of the biscuits to minimize the firm’s cost?

- a) Formulate this as a linear programming problem.
- b) Solve this problem graphically, giving the optimal values of all variables.
- c) What is the total cost of a package of dog biscuits using the optimal mix? **Px**

• **B.28** The Sweet Smell Fertilizer Company markets bags of manure labeled “not less than 60 lb dry weight.” The packaged manure is a combination of compost and sewage wastes. To provide good-quality fertilizer, each bag should contain at least 30 lb of compost but no more than 40 lb of sewage. Each pound of compost costs Sweet Smell 5¢ and each pound of sewage costs 4¢. Use a graphical LP method to determine the least-cost blend of compost and sewage in each bag. **Px**

• **B.29** Consider Paul Jordan’s following linear programming formulation:

$$\begin{aligned} \text{Minimize cost} &= \$1X_1 + \$2X_2 \\ \text{Subject to:} & \quad X_1 + 3X_2 \geq 90 \\ & \quad 8X_1 + 2X_2 \geq 160 \\ & \quad 3X_1 + 2X_2 \geq 120 \\ & \quad X_2 \leq 70 \end{aligned}$$

- a) Graphically illustrate the feasible region to indicate to Jordan which corner point produces the optimal solution.
- b) What is the cost of this solution? **Px**

• **B.30** Solve the following linear programming problem graphically:

$$\begin{aligned} \text{Minimize cost} &= 4X_1 + 5X_2 \\ \text{Subject to:} & \quad X_1 + 2X_2 \geq 80 \\ & \quad 3X_1 + X_2 \geq 75 \\ & \quad X_1, X_2 \geq 0 \end{aligned} \quad \mathbf{Px}$$

•• **B.31** How many corner points are there in the feasible region of the following problem?

$$\begin{aligned} \text{Minimize cost} &= X - Y \\ \text{Subject to:} & \quad X \leq 4 \\ & \quad -X \leq 2 \\ & \quad X + 2Y \leq 6 \\ & \quad -X + 2Y \leq 8 \\ & \quad Y \geq 0 \end{aligned}$$

(Note: X values can be negative in this problem.)

Additional problems **B.32–B.33** are available in MyOMLab.

**Problems B.34–B.40** relate to Linear Programming Applications

••• **B.34** The Hills County, Michigan, superintendent of education is responsible for assigning students to the three high schools in his county. He recognizes the need to bus a certain number of students, because several sectors, A–E, of the county are beyond walking distance to a school. The superintendent partitions the county into five geographic sectors as he attempts to establish a plan that will minimize the total number of student miles traveled by bus. He also recognizes that if a student happens to live in a certain sector and is assigned to the high school in that sector, there is no need to bus him because he can walk to school. The three schools are located in sectors B, C, and E.



Hans Magelssen/Shutterstock

The accompanying table reflects the number of high-school-age students living in each sector and the distance in miles from each sector to each school:

DISTANCE TO SCHOOL				
SECTOR	SCHOOL IN SECTOR B	SCHOOL IN SECTOR C	SCHOOL IN SECTOR E	NUMBER OF STUDENTS
A	5	8	6	700
B	0	4	12	500
C	4	0	7	100
D	7	2	5	800
E	12	7	0	400
				2,500

Each high school has a capacity of 900 students.

- a) Set up the objective function and constraints of this problem using linear programming so that the total number of student miles traveled by bus is minimized.
- b) Solve the problem. **Px**

•• **B.35** The Rio Credit Union has \$250,000 available to invest in a 12-month commitment. The money can be placed in Brazilian treasury notes yielding an 8% return or in riskier high-yield bonds at an average rate of return of 9%. Credit union regulations require diversification to the extent that at least 50% of the investment be placed in Treasury notes. It is also decided that no more than 40% of the investment be placed in bonds. How much should the Rio Credit Union invest in each security so as to maximize its return on investment? **Px**

•• **B.36** Wichita’s famous Sethi Restaurant is open 24 hours a day. Servers report for duty at 3 A.M., 7 A.M., 11 A.M., 3 P.M., 7 P.M., or 11 P.M., and each works an 8-hour shift. The following table shows the minimum number of workers needed during the 6 periods into which the day is divided:

PERIOD	TIME	NUMBER OF SERVERS REQUIRED
1	3 A.M.–7 A.M.	3
2	7 A.M.–11 A.M.	12
3	11 A.M.–3 P.M.	16
4	3 P.M.–7 P.M.	9
5	7 P.M.–11 P.M.	11
6	11 P.M.–3 A.M.	4

Owner Avanti Sethi’s scheduling problem is to determine how many servers should report for work at the start of each time period in order to minimize the total staff required for one day’s operation. (Hint: Let  $X_i$  equal the number of servers beginning work in time period  $i$ , where  $i = 1, 2, 3, 4, 5, 6$ .) **Px**

•• **B.37** Leach Distributors packages and distributes industrial supplies. A standard shipment can be packaged in a class A container, a class K container, or a class T container. A single class A container yields a profit of \$9; a class K container, a profit of \$7; and a class T container, a profit of \$15. Each shipment prepared requires a certain amount of packing material and a certain amount of time.

RESOURCES NEEDED PER STANDARD SHIPMENT		
CLASS OF CONTAINER	PACKING MATERIAL (POUNDS)	PACKING TIME (HOURS)
A	2	2
K	1	6
T	3	4

Total resource available each week: 130 pounds 240 hours

Hugh Leach, head of the firm, must decide the optimal number of each class of container to pack each week. He is bound by the previously mentioned resource restrictions but also decides that he must keep his 6 full-time packers employed all 240 hours (6 workers × 40 hours) each week.

Formulate and solve this problem using LP software. **Px**

•••**B.38** Tri-State Manufacturing has three factories (1, 2, and 3) and three warehouses (A, B, and C). The following table shows the shipping costs between each factory and warehouse, the factory manufacturing capabilities (in thousands), and the warehouse capacities (in thousands). Management would like to keep the warehouses filled to capacity in order to generate demand.

TO \ FROM	WAREHOUSE A	WAREHOUSE B	WAREHOUSE C	PRODUCTION CAPABILITY
Factory 1	\$ 6	\$ 5	\$ 3	6
Factory 2	\$ 8	\$10	\$ 8	8
Factory 3	\$11	\$14	\$18	10
Capacity	7	12	5	

- a) Write the objective function and the constraint equations. Let  $X_{1A}$  = 1,000s of units shipped from factory 1 to warehouse A, and so on.
- b) Solve by computer. **Px**

•••• **B.39** Bowman Builders manufactures steel storage sheds for commercial use. Joe Bowman, president of Bowman Builders, is contemplating producing sheds for home use. The activities necessary to build an experimental model and related data are given in Table B.2.

- a) What is the project normal time completion date? (See Chapter 3 for a review of project management.)
- b) Formulate an LP problem to crash this project to 10 weeks.

**TABLE B.2** Data for Problem B.39

ACTIVITY	NORMAL TIME	CRASH TIME	NORMAL COST (\$)	CRASH COST (\$)	IMMEDIATE PREDECESSORS
A	3	2	1,000	1,600	—
B	2	1	2,000	2,700	—
C	1	1	300	300	—
D	7	3	1,300	1,600	A
E	6	3	850	1,000	B
F	2	1	4,000	5,000	C
G	4	2	1,500	2,000	D, E

•••• **B.40** You have just been hired as a planner for the municipal school system, and your first assignment is to redesign the subsidized lunch program. In particular, you are to formulate the least expensive lunch menu that will still meet all state and federal nutritional guidelines.

The guidelines are as follows: A meal must be between 500 and 800 calories. It must contain at least 200 calories of protein, at least 200 calories of carbohydrates, and no more than 400 calories of fat. It also needs to have at least 200 calories of a food classified as a fruit or vegetable.

Table B.3 provides a list of the foods you can consider as possible menu items, with contract-determined prices and nutritional information. Note that all percentages sum to 100% per food—as all calories are protein, carbohydrate, or fat calories. For example, a serving of applesauce has 100 calories, all of which are carbohydrates, and it counts as a fruit/veg food. You are allowed to use fractional servings, such as 2.25 servings of turkey breast and a 0.33 portion of salad. Costs and nutritional attributes scale likewise: e.g., a 0.33 portion of salad costs \$.30 and has 33 calories.

**TABLE B.3** Data for Problem B.40

FOOD	COST/SERVING	CALORIES/SERVING	% PROTEIN	% CARBS	% FAT	FRUIT/VEG
Applesauce	\$0.30	100	0%	100%	0%	Y
Canned corn	\$0.40	150	20%	80%	0%	Y
Fried chicken	\$0.90	250	55%	5%	40%	N
French fries	\$0.20	400	5%	35%	60%	N
Mac and cheese	\$0.50	430	20%	30%	50%	N
Turkey breast	\$1.50	300	67%	0%	33%	N
Garden salad	\$0.90	100	15%	40%	45%	Y

Formulate and solve as a linear problem. Print out your formulation in Excel showing the objective function coefficients and constraint matrix in standard form.

- ◆ Display, on a separate page, the full *Answer Report* as generated by Excel Solver.
- ◆ Highlight and label as *Z* the objective value for the optimal solution on the *Answer Report*.
- ◆ Highlight the nonzero decision variables for the optimal solution on the *Answer Report*.
- ◆ Display, on a separate page, the full *Sensitivity Report* as generated by Excel Solver. **Px**

**Problems B.41–B.42 relate to Integer and Binary Variables**

•• **B.41** Rollins Publishing needs to decide what textbooks from the following table to publish.

TEXT-BOOK	DEMAND	FIXED COST	VARIABLE COST	SELLING PRICE
Book 1	9,000	\$12,000	\$19	\$40
Book 2	8,000	\$21,000	\$28	\$60
Book 3	5,000	\$15,000	\$30	\$52
Book 4	6,000	\$10,000	\$20	\$34
Book 5	7,000	\$18,000	\$20	\$45

For each book, the maximum demand, fixed cost of publishing, variable cost, and selling price are provided. Rollins has the capacity to publish a total of 20,000 books.

- a) Formulate this problem to determine which books should be selected and how many of each should be published to maximize profit.
- b) Solve using computer software. **Px**

•• **B.42** Porter Investments needs to develop an investment portfolio for Mrs. Singh from the following list of possible investments:

INVESTMENT	COST	EXPECTED RETURN
A	\$10,000	\$ 700
B	\$12,000	\$1,000
C	\$ 3,500	\$ 390
D	\$ 5,000	\$ 500
E	\$ 8,500	\$ 750
F	\$ 8,000	\$ 640
G	\$ 4,000	\$ 300

Mrs. Singh has a total of \$60,000 to invest. The following conditions must be met: (1) If investment F is chosen, then investment G must also be part of the portfolio, (2) at least four investments should be chosen, and (3) of investments A and B, exactly one must be included. Formulate and solve this problem using LP software to determine which stocks should be included in Mrs. Singh’s portfolio. **Px**

## CASE STUDIES

### Quain Lawn and Garden, Inc.

Bill and Jeanne Quain spent a career as a husband-and-wife real estate investment partnership in Atlantic City, New Jersey. When they finally retired to a 25-acre farm in nearby Cape May County, they became ardent amateur gardeners. Bill planted shrubs and fruit trees, and Jeanne spent her hours potting all sizes of plants. When the volume of shrubs and plants reached the point that the Quains began to think of their hobby in a serious vein, they built a greenhouse adjacent to their home and installed heating and watering systems.

By 2012, the Quains realized their retirement from real estate had really only led to a second career—in the plant and shrub business—and they filed for a New Jersey business license. Within

a matter of months, they asked their attorney to file incorporation documents and formed the firm Quain Lawn and Garden, Inc.

Early in the new business’s existence, Bill Quain recognized the need for a high-quality commercial fertilizer that he could blend himself, both for sale and for his own nursery. His goal was to keep his costs to a minimum while producing a top-notch product that was especially suited to the New Jersey climate.

Working with chemists at Rutgers University, Quain blended “Quain-Grow.” It consists of four chemical compounds, C-30, C-92, D-21, and E-11. The cost per pound for each compound is indicated in the table on the next page:

CHEMICAL COMPOUND	COST PER POUND
C-30	\$.12
C-92	.09
D-21	.11
E-11	.04

The specifications for Quain-Grow are established as:

- Chemical E-11 must constitute at least 15% of the blend.
- C-92 and C-30 must together constitute at least 45% of the blend.

- D-21 and C-92 can together constitute no more than 30% of the blend.
- Quain-Grow is packaged and sold in 50-lb bags.

### Discussion Questions

- Formulate an LP problem to determine what blend of the four chemicals will allow Quain to minimize the cost of a 50-lb bag of the fertilizer.
- Solve to find the best solution.

## Scheduling Challenges at Alaska Airlines

Good airline scheduling is essential to delivering outstanding customer service with high plane utilization rates. Airlines must schedule pilots, flight attendants, aircraft, baggage handlers, customer service agents, and ramp crews. At Alaska Airlines, it all begins with seasonal flight schedules that are developed 330 days in advance.

Revenue and marketing goals drive the potential routing decisions, but thousands of constraints impact these schedules. Using SABRE scheduling optimizer software, Alaska considers the number of planes available, seat capacity, ranges, crew availability, union contracts that dictate hours that crews can fly, and maintenance regulations that regularly take planes out of service, just to name a few. Alaska's scheduling department sends preliminary schedules to the human resources, maintenance, operations, customer service, marketing, and other departments for feedback before finalizing flight schedules.

Alaska Airlines' historic mission is to serve its extremely loyal customer base in the remote and unreachable small towns in Alaska. Serving many airports in Alaska is especially complex because the airline requires its pilots to have special skills to deal with extremely adverse weather, tight mountain passes, and short runways. Some airports lack full-time TSA agents or strong ground support and may not even be open 24 hours per day. In some cases, runways are not plowed because the village plow is busy clearing the roads for school buses. Navigational technology developed by Alaska Airlines has significantly reduced weather-related cancelled flights as Alaska can now land where many other carriers cannot.

After the SABRE optimizer schedules thousands of flights, scheduling activity turns to the next step: crew optimizing. The crew optimizer (Alaska uses Jeppesen software developed by Boeing and based on linear programming) attempts to eliminate unnecessary layovers and crew idle time while adhering to FAA and union restrictions. Alaska leads the industry in pilot "hard time" (i.e., the amount of time a pilot is being paid when passengers are actually being moved). After the crew requirements for every flight are determined, the 3,000 flight attendants and 1,500 pilots rank their preferred routings on a monthly basis. Personnel are assigned to each flight using seniority and feasibility.

## Video Case

Interestingly, not every pilot or flight attendant always bids on the Hawaii routes (about 20% of all flights), the long-haul East Coast routes, or the Mexico flights. Some prefer the flying challenge of the "milk run" flights to Ketchikan, Sitka, Juneau, Fairbanks, Anchorage, and back to Seattle, which are in keeping with the culture and contact with local residents.

As an airline that accentuates risk taking and empowers employees to think "out of the box," Alaska recently decided to experiment with a schedule change on its Seattle-to-Chicago route. Given crew restrictions on flying hours per day, the flight had previously included a crew layover in Chicago. When a company analyst documented the feasibility of running the same crew on the two 4-hour legs of the round trip (which implied an *extremely* tight turnaround schedule in Chicago), his data indicated that on 98.7% of the round trip flights, the crew would not "time out." His boss gave the go-ahead.

### Discussion Questions\*

- Why is scheduling for Alaska more complex than for other airlines?
- What operational considerations may prohibit Alaska from adding flights and more cities to its network?
- What were the risks of keeping the same crew on the Seattle—Chicago—Seattle route?
- Estimate the direct costs to the airline should the crew "time out" and not be able to fly its Boeing 737 back to Seattle from Chicago on the same day. These direct variable costs should include moving and parking the plane overnight along with hotel and meal costs for the crew and passengers. Do you think this is more advantageous than keeping a spare crew in Chicago?

\*You may wish to view the video that accompanies this case before addressing these questions.

- Additional Case Studies:** Visit [MyOMLab](#) for these free case studies:

**Chase Manhattan Bank:** This scheduling case involves finding the optimal number of full-time versus part-time employees at a bank.

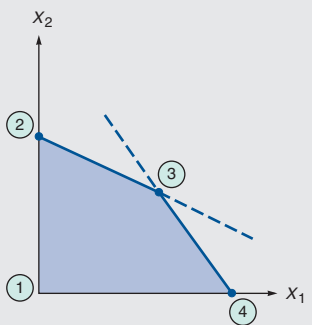
**Coastal States Chemical:** The company must prepare for a shortage of natural gas.

## Endnotes

- Iso* means "equal" or "similar." Thus, an iso-profit line represents a line with all profits the same, in this case \$210.
- See, for example, Barry Render, Ralph M. Stair, Michael Hanna, and T. Hale, *Quantitative Analysis for Management*, 12th ed. (Pearson Education, Inc., Upper Saddle River, NJ, 2013):

Chapters 7–9; or Raju Balakrishnan, Barry Render, and Ralph M. Stair, *Managerial Decision Modeling with Spreadsheets*, 3rd ed. (Pearson Education, Inc., Upper Saddle River, NJ, 2012): Chapters 2–4.

# Module B *Rapid Review*

Main Heading	Review Material	MyOMLab
<b>WHY USE LINEAR PROGRAMMING?</b> (p. 700)	<ul style="list-style-type: none"> <li>■ <b>Linear programming (LP)</b>—A mathematical technique designed to help operations managers plan and make decisions relative to allocation of resources.</li> </ul>	Concept Questions: 1.1–1.4 <b>VIDEO B.1</b> Scheduling Challenges at Alaska Airlines
<b>REQUIREMENTS OF A LINEAR PROGRAMMING PROBLEM</b> (p. 701)	<ul style="list-style-type: none"> <li>■ <b>Objective function</b>—A mathematical expression in linear programming that maximizes or minimizes some quantity (often profit or cost, but any goal may be used).</li> <li>■ <b>Constraints</b>—Restrictions that limit the degree to which a manager can pursue an objective.</li> </ul> <p>All LP problems have four properties in common:</p> <ol style="list-style-type: none"> <li>1. LP problems seek to <i>maximize</i> or <i>minimize</i> some quantity. We refer to this property as the <i>objective function</i> of an LP problem.</li> <li>2. The presence of restrictions, or <i>constraints</i>, limits the degree to which we can pursue our objective. We want, therefore, to maximize or minimize a quantity (the objective function) subject to limited resources (the constraints).</li> <li>3. There must be <i>alternative courses of action</i> to choose from.</li> <li>4. The objective and constraints in linear programming problems must be expressed in terms of <i>linear equations</i> or inequalities.</li> </ol>	Concept Questions: 2.1–2.4 Problem: B.1
<b>FORMULATING LINEAR PROGRAMMING PROBLEMS</b> (pp. 701–702)	<p>One of the most common linear programming applications is the <i>product-mix problem</i>. Two or more products are usually produced using limited resources. For example, a company might like to determine how many units of each product it should produce to maximize overall profit, given its limited resources.</p> <p>An important aspect of linear programming is that certain interactions will exist between variables. The more units of one product that a firm produces, the fewer it can make of other products.</p>	Concept Questions: 3.1–3.4 Virtual Office Hours for Solved Problem: B.1 <b>ACTIVE MODEL B.1</b>
<b>GRAPHICAL SOLUTION TO A LINEAR PROGRAMMING PROBLEM</b> (pp. 702–705)	<ul style="list-style-type: none"> <li>■ <b>Graphical solution approach</b>—A means of plotting a solution to a two-variable problem on a graph.</li> <li>■ <b>Decision variables</b>—Choices available to a decision maker.</li> </ul> <p>Constraints of the form <math>X \geq 0</math> are called <i>nonnegativity constraints</i>.</p> <ul style="list-style-type: none"> <li>■ <b>Feasible region</b>—The set of all feasible combinations of decision variables. Any point inside the feasible region represents a <i>feasible solution</i>, while any point outside the feasible region represents an <i>infeasible solution</i>.</li> <li>■ <b>Iso-profit line method</b>—An approach to identifying the optimum point in a graphic linear programming problem. The line that touches a particular point of the feasible region will pinpoint the optimal solution.</li> <li>■ <b>Corner-point method</b>—Another method for solving graphical linear programming problems.</li> </ul> <p>The mathematical theory behind linear programming states that an optimal solution to any problem will lie at a <i>corner point</i>, or an <i>extreme point</i>, of the feasible region. Hence, it is necessary to find only the values of the variables at each corner; the optimal solution will lie at one (or more) of them. This is the corner-point method.</p> 	Concept Questions: 4.1–4.4 Problems: B.2–B.21 Virtual Office Hours for Solved Problem: B.2
<b>SENSITIVITY ANALYSIS</b> (pp. 705–708)	<ul style="list-style-type: none"> <li>■ <b>Parameter</b>—A numerical value that is given in a model.</li> <li>■ <b>Sensitivity analysis</b>—An analysis that projects how much a solution may change if there are changes in the variables or input data.</li> </ul> <p>Sensitivity analysis is also called <i>postoptimality analysis</i>. There are two approaches to determining just how sensitive an optimal solution is to changes: (1) a trial-and-error approach and (2) the analytic postoptimality method.</p>	Concept Questions: 5.1–5.4 Problems: B.22–B.24

Main Heading	Review Material	
	<p>To use the analytic postoptimality method, after an LP problem has been solved, we determine a range of changes in problem parameters that will not affect the optimal solution or change the variables in the solution. LP software has this capability.</p> <p>While using the information in a sensitivity report to answer what-if questions, we assume that we are considering a change to only a <i>single</i> input data value at a time. That is, the sensitivity information does not generally apply to simultaneous changes in several input data values.</p> <ul style="list-style-type: none"> <li>▪ <b>Shadow price (or dual value)</b>—The value of one additional unit of a scarce resource in LP.</li> </ul> <p>The shadow price is valid as long as the right-hand side of the constraint stays in a range within which all current corner points continue to exist. The information to compute the upper and lower limits of this range is given by the entries labeled Allowable Increase and Allowable Decrease in the sensitivity report.</p>	
<b>SOLVING MINIMIZATION PROBLEMS</b> (pp. 708–709)	<ul style="list-style-type: none"> <li>▪ <b>Iso-cost</b>—An approach to solving a linear programming minimization problem graphically.</li> </ul> <p>The iso-cost line approach to solving minimization problems is analogous to the iso-profit approach for maximization problems, but successive iso-cost lines are drawn <i>inward</i> instead of outward.</p>	<p>Concept Questions: 6.1–6.3</p> <p>Problems: B.25–B.33</p> <p>Virtual Office Hours for Solved Problem: B.3</p>
<b>LINEAR PROGRAMMING APPLICATIONS</b> (pp. 710–713)	<p>The <i>diet problem</i>, known in agricultural applications as the <i>feed-mix problem</i>, involves specifying a food or feed ingredient combination that will satisfy stated nutritional requirements at a minimum cost level.</p> <p><i>Labor scheduling problems</i> address staffing needs over a specific time period. They are especially useful when managers have some flexibility in assigning workers to jobs that require overlapping or interchangeable talents.</p>	<p>Concept Questions: 7.1–7.3</p> <p>Problems: B.34–B.40</p>
<b>THE SIMPLEX METHOD OF LP</b> (p. 713)	<ul style="list-style-type: none"> <li>▪ <b>Simplex method</b>—An algorithm for solving linear programming problems of all sizes.</li> </ul> <p>The simplex method is actually a set of instructions with which we examine corner points in a methodical fashion until we arrive at the best solution—highest profit or lowest cost. Computer programs (such as Excel OM and POM for Windows) and Excel’s Solver add-in are available to solve linear programming problems via the simplex method.</p>	<p>Concept Questions: 8.1–8.2</p> <p>Virtual Office Hours for Solved Problem: C.1 (note that this Module C video is an LP application of the transportation problem)</p>
<b>INTEGER AND BINARY VARIABLES</b> (pp. 713–715)	<ul style="list-style-type: none"> <li>▪ <b>Binary variables</b>—Decision variables that can only take on the value of 0 or 1.</li> </ul> <p>Using computer software, decision variables for linear programs can be forced to be integer or even binary. Binary variables extend the flexibility of linear programs to include such options as mutually exclusive alternatives, either-or constraints, contingent decisions, fixed-charge problems, and threshold levels.</p>	<p>Concept Questions: 9.1–9.4</p> <p>Problems: B.41–B.42</p>

## Self Test

▪ **Before taking the self-test**, refer to the learning objectives listed at the beginning of the module and the key terms listed at the end of the module.

- LO B.1** Which of the following is *not* a valid LP constraint formulation?
- a)  $3X + 4Y \leq 12$       b)  $2X \times 2Y \leq 12$   
 c)  $3Y + 2Z = 18$       d)  $100 \geq X + Y$   
 e)  $2.5X + 1.5Z = 30.6$
- LO B.2** Using a *graphical solution procedure* to solve a maximization problem requires that we:
- a) move the iso-profit line up until it no longer intersects with any constraint equation.  
 b) move the iso-profit line down until it no longer intersects with any constraint equation.  
 c) apply the method of simultaneous equations to solve for the intersections of constraints.  
 d) find the value of the objective function at the origin.
- LO B.3** Consider the following linear programming problem:  
 Maximize  $4X + 10Y$   
 Subject to:  $3X + 4Y \leq 480$   
 $4X + 2Y \leq 360$   
 $X, Y \geq 0$
- The feasible corner points are (48,84), (0,120), (0,0), and (90,0). What is the maximum possible value for the objective function?  
 a) 1,032      b) 1,200      c) 360      d) 1,600      e) 840
- LO B.4** A zero shadow price for a resource ordinarily means that:
- a) the resource is scarce.  
 b) the resource constraint was redundant.  
 c) the resource has not been used up.  
 d) something is wrong with the problem formulation.  
 e) none of the above.
- LO B.5** For these two constraints, which point is in the feasible region of this minimization problem?
- $$14x + 6y \geq 42 \quad \text{and} \quad x + y \geq 3$$
- a)  $x = -1, y = 1$       b)  $x = 0, y = 4$   
 c)  $x = 2, y = 1$       d)  $x = 5, y = 1$   
 e)  $x = 2, y = 0$
- LO B.6** When applying LP to diet problems, the objective function is usually designed to:
- a) maximize profits from blends of nutrients.  
 b) maximize ingredient blends.  
 c) minimize production losses.  
 d) maximize the number of products to be produced.  
 e) minimize the costs of nutrient blends.

Answers: LO B.1. b; LO B.2. a; LO B.3. b; LO B.4. c; LO B.5. d; LO B.6. e.

# Transportation Models

C  
MODULE

## MODULE OUTLINE

- ◆ Transportation Modeling 730
- ◆ Developing an Initial Solution 732
- ◆ The Stepping-Stone Method 734
- ◆ Special Issues in Modeling 737



Alaska Airlines



Alaska Airlines

# LEARNING OBJECTIVES

## LO C.1

**Develop** an initial solution to a transportation model with the northwest-corner and intuitive lowest-cost methods 732

## LO C.2

**Solve** a problem with the stepping-stone method 734

## LO C.3

**Balance** a transportation problem 737

## LO C.4

**Deal** with a problem that has degeneracy 737

The problem facing rental companies like Avis, Hertz, and National is cross-country travel. Lots of it. Cars rented in New York end up in Chicago, cars from L.A. come to Philadelphia, and cars from Boston come to Miami. The scene is repeated in over 100 cities around the U.S. As a result, there are too many cars in some cities and too few in others. Operations managers have to decide how many of these rentals should be trucked (by costly auto carriers) from each city with excess capacity to each city that needs more rentals. The process requires quick action for the most economical routing, so rental car companies turn to transportation modeling.



Vibrant Image Studio/Shutterstock

## Transportation Modeling

Because location of a new factory, warehouse, or distribution center is a strategic issue with substantial cost implications, most companies consider and evaluate several locations. With a wide variety of objective and subjective factors to be considered, rational decisions are aided by a number of techniques. One of those techniques is transportation modeling.

The transportation models described in this module prove useful when considering alternative facility locations *within the framework of an existing distribution system*. Each new potential plant, warehouse, or distribution center will require a different allocation of shipments, depending on its own production and shipping costs and the costs of each existing facility. The choice of a new location depends on which will yield the minimum cost *for the entire system*.

**Transportation modeling** finds the least-cost means of shipping supplies from several origins to several destinations. *Origin points* (or *sources*) can be factories, warehouses, car rental agencies like Avis, or any other points from which goods are shipped. *Destinations* are any points that receive goods. To use the transportation model, we need to know the following:

1. The origin points and the capacity or supply per period at each.
2. The destination points and the demand per period at each.
3. The cost of shipping one unit from each origin to each destination.

The transportation model is one form of the linear programming models discussed in Business Analytics Module B. Software is available to solve both transportation problems and the more general class of linear programming problems. To fully use such programs, though, you need to understand the assumptions that underlie the model. To illustrate the transportation problem, we now look at a company called Arizona Plumbing, which makes, among other products,

### Transportation modeling

An iterative procedure for solving problems that involves minimizing the cost of shipping products from a series of sources to a series of destinations.



**TABLE C.1** Transportation Costs Per Bathtub for Arizona Plumbing

FROM \ TO	ALBUQUERQUE	BOSTON	CLEVELAND
Des Moines	\$5	\$4	\$3
Evansville	\$8	\$4	\$3
Fort Lauderdale	\$9	\$7	\$5

a full line of bathtubs. In our example, the firm must decide which of its factories should supply which of its warehouses. Relevant data for Arizona Plumbing are presented in Table C.1 and Figure C.1. Table C.1 shows, for example, that it costs Arizona Plumbing \$5 to ship one bathtub from its Des Moines factory to its Albuquerque warehouse, \$4 to Boston, and \$3 to Cleveland.

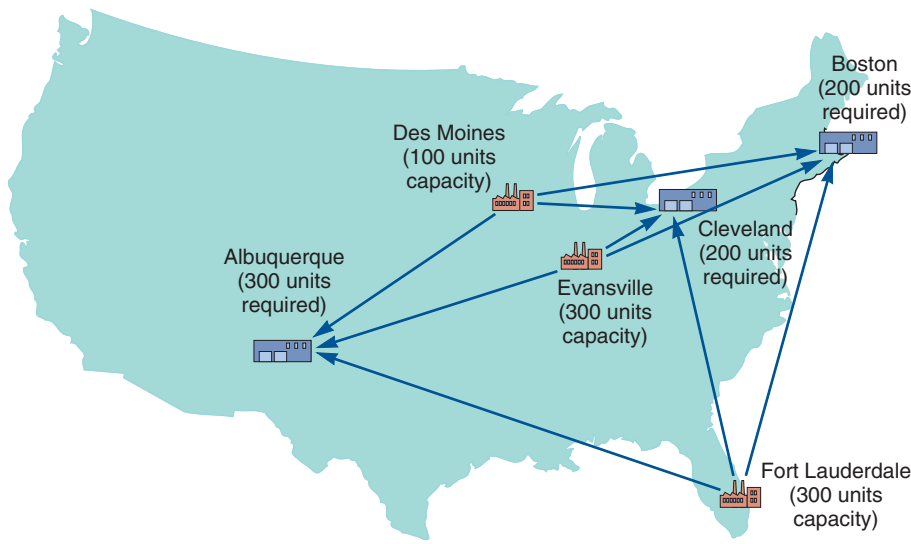


Figure C.1  
Transportation Problem

Likewise, we see in Figure C.1 that the 300 units required by Arizona Plumbing’s Albuquerque warehouse may be shipped in various combinations from its Des Moines, Evansville, and Fort Lauderdale factories.

The first step in the modeling process is to set up a *transportation matrix*. Its purpose is to summarize all relevant data and to keep track of algorithm computations. Using the information displayed in Figure C.1 and Table C.1, we can construct a transportation matrix as shown in Figure C.2.

From \ To	Albuquerque	Boston	Cleveland	Factory capacity
Des Moines	\$5	\$4	\$3	100
Evansville	\$8	\$4	\$3	300
Fort Lauderdale	\$9	\$7	\$5	300
Warehouse requirement	300	200	200	700

Des Moines capacity constraint  
 Cell representing a possible source-to-destination shipping assignment (Evansville to Cleveland)  
 Cost of shipping 1 unit from Fort Lauderdale factory to Boston warehouse  
 Cleveland warehouse demand  
 Total demand and total supply

Figure C.2  
Transportation Matrix for Arizona Plumbing

**LO C.1** Develop an initial solution to a transportation model with the northwest-corner and intuitive lowest-cost methods

# Developing an Initial Solution

Once the data are arranged in tabular form, we must establish an initial feasible solution to the problem. A number of different methods have been developed for this step. We now discuss two of them, the northwest-corner rule and the intuitive lowest-cost method.

## The Northwest-Corner Rule

The **northwest-corner rule** requires that we start in the upper-left-hand cell (or northwest corner) of the table and allocate units to shipping routes as follows:

1. Exhaust the supply (factory capacity) of each row (e.g., Des Moines: 100) before moving down to the next row.
2. Exhaust the (warehouse) requirements of each column (e.g., Albuquerque: 300) before moving to the next column on the right.
3. Check to ensure that all supplies and demands are met.

Example C1 applies the northwest-corner rule to our Arizona Plumbing problem.

### Example C1

#### THE NORTHWEST-CORNER RULE

Arizona Plumbing wants to use the northwest-corner rule to find an initial solution to its problem.

**APPROACH** ▶ Follow the three steps listed above. See Figure C.3.

**SOLUTION** ▶ To make the initial solution, these five assignments are made:

1. Assign 100 tubs from Des Moines to Albuquerque (exhausting Des Moines's supply).
2. Assign 200 tubs from Evansville to Albuquerque (exhausting Albuquerque's demand).
3. Assign 100 tubs from Evansville to Boston (exhausting Evansville's supply).
4. Assign 100 tubs from Fort Lauderdale to Boston (exhausting Boston's demand).
5. Assign 200 tubs from Fort Lauderdale to Cleveland (exhausting Cleveland's demand and Fort Lauderdale's supply).

Figure C.3

**Northwest-Corner Solution to Arizona Plumbing Problem**

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	100 \$5			100
(E) Evansville	200 \$8	100 \$4		300
(F) Fort Lauderdale		100 \$7	200 \$5	300
Warehouse requirement	300	200	200	700

Means that the firm is shipping 100 bathtubs from Fort Lauderdale to Boston

The total cost of this shipping assignment is \$4,200 (see Table C.2).

**TABLE C.2** Computed Shipping Cost

ROUTE				
FROM	TO	TUBS SHIPPED	COST PER UNIT	TOTAL COST
D	A	100	\$5	\$ 500
E	A	200	8	1,600
E	B	100	4	400
F	B	100	7	700
F	C	200	5	\$1,000
				Total: \$4,200

**INSIGHTS** ► The solution given is feasible because it satisfies all demand and supply constraints. The northwest-corner rule is easy to use, but it totally ignores costs, and therefore should only be considered as a starting position.

**LEARNING EXERCISE** ► Does the shipping assignment change if the cost from Des Moines to Albuquerque increases from \$5 per unit to \$10 per unit? Does the total cost change? [Answer: The initial assignment is the same, but cost = \$4,700.]

**RELATED PROBLEMS** ► C.1a, C.3a, C.15

## The Intuitive Lowest-Cost Method

The **intuitive method** makes initial allocations based on lowest cost. This straightforward approach uses the following steps:

1. Identify the cell with the lowest cost. Break any ties for the lowest cost arbitrarily.
2. Allocate as many units as possible to that cell without exceeding the supply or demand. Then cross out that row or column (or both) that is exhausted by this assignment.
3. Find the cell with the lowest cost from the remaining (not crossed out) cells.
4. Repeat Steps 2 and 3 until all units have been allocated.

### Intuitive method

A cost-based approach to finding an initial solution to a transportation problem.

## Example C2

### THE INTUITIVE LOWEST-COST APPROACH

Arizona Plumbing now wants to apply the intuitive lowest-cost approach.

**APPROACH** ► Apply the 4 steps listed above to the data in Figure C.2.

**SOLUTION** ► When the firm uses the intuitive approach on the data (rather than the northwest-corner rule) for its starting position, it obtains the solution seen in Figure C.4.

The total cost of this approach =  $\$3(100) + \$3(100) + \$4(200) + \$9(300) = \$4,100$ .  
(D to C) (E to C) (E to B) (F to A)

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	\$5	\$4	\$3	100
(E) Evansville	\$8	\$4	\$3	300
(F) Fort Lauderdale	\$9	\$7	\$5	300
Warehouse requirement	300	200	200	700

First, cross out top row (D) after entering 100 units in \$3 cell because row D is satisfied.  
 Second, cross out column C after entering 100 units in this \$3 cell because column C is satisfied.  
 Third, cross out row E and column B after entering 200 units in this \$4 cell because a total of 300 units satisfies row E and column B.  
 Finally, enter 300 units in the only remaining cell to complete the allocations.

Figure C.4

### Intuitive Lowest-Cost Solution to Arizona Plumbing Problem

**INSIGHT** ► This method's name is appropriate as most people find it intuitively correct to include costs when making an initial assignment.

**LEARNING EXERCISE** ► If the cost per unit from Des Moines to Cleveland is not \$3, but rather \$6, does this initial solution change? [Answer: Yes, now D - B = 100, D - C = 0, E - B = 100, E - C = 200, F - A = 300. Others unchanged at zero. Total cost stays the same.]

**RELATED PROBLEMS** ► C.1b, C.2, C.3b

Although the likelihood of a minimum-cost solution *does* improve with the intuitive method, we would have been fortunate if the intuitive solution yielded the minimum cost. In this case, as in the northwest-corner solution, it did not. Because the northwest-corner and the intuitive

lowest-cost approaches are meant only to provide us with a starting point, we often will have to employ an additional procedure to reach an *optimal* solution.

## The Stepping-Stone Method

### Stepping-stone method

An iterative technique for moving from an initial feasible solution to an optimal solution in the transportation method.

**LO C.2** Solve a problem with the stepping-stone method

The **stepping-stone method** will help us move from an initial feasible solution to an optimal solution. It is used to evaluate the cost effectiveness of shipping goods via transportation routes not currently in the solution. When applying it, we test each unused cell, or square, in the transportation table by asking: What would happen to total shipping costs if one unit of the product (for example, one bathtub) was tentatively shipped on an unused route? We conduct the test as follows:

1. Select any unused square to evaluate.
2. Beginning at this square, trace a closed path back to the original square via squares that are currently being used (only horizontal and vertical moves are permissible). You may, however, step over either an empty or an occupied square.
3. Beginning with a plus (+) sign at the unused square, place alternating minus signs and plus signs on each corner square of the closed path just traced.
4. Calculate an improvement index by first adding the unit-cost figures found in each square containing a plus sign and then by subtracting the unit costs in each square containing a minus sign.
5. Repeat Steps 1 through 4 until you have calculated an improvement index for all unused squares. If all indices computed are *greater than or equal to zero*, you have reached an optimal solution. If not, the current solution can be improved further to decrease total shipping costs.

Example C3 illustrates how to use the stepping-stone method to move toward an optimal solution. We begin with the northwest-corner initial solution developed in Example C1.

### Example C3

#### CHECKING UNUSED ROUTES WITH THE STEPPING-STONE METHOD

Arizona Plumbing wants to evaluate unused shipping routes.

**APPROACH** ► Start with Example C1's Figure C.3 and follow the 5 steps listed above. As you can see, the four currently unassigned routes are Des Moines to Boston, Des Moines to Cleveland, Evansville to Cleveland, and Fort Lauderdale to Albuquerque.

**SOLUTION** ► **Steps 1 and 2.** Beginning with the Des Moines–Boston route, trace a closed path *using only currently occupied squares* (see Figure C.5). Place alternating plus and minus signs in the corners of this path. In the upper-left square, for example, we place a minus sign because we have *subtracted* 1 unit from the original 100. Note that we can use only squares currently used for shipping to turn the corners of the route we are tracing. Hence, the path Des Moines–Boston to Des Moines–Albuquerque to Fort Lauderdale–Albuquerque to Fort Lauderdale–Boston to Des Moines–Boston would not be acceptable because the Fort Lauderdale–Albuquerque square is empty. It turns out that *only one closed route exists for each empty square*. Once this one closed path is identified, we can begin assigning plus and minus signs to these squares in the path.

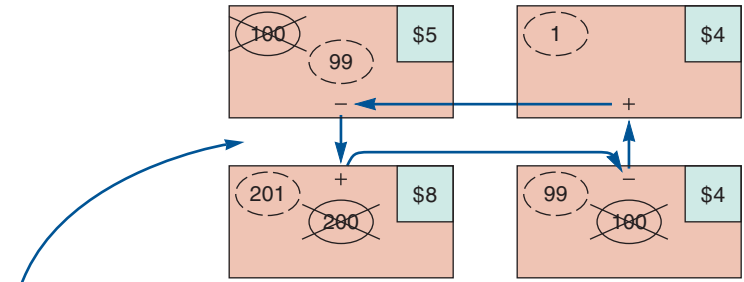
**Step 3.** How do we decide which squares get plus signs and which squares get minus signs? The answer is simple. Because we are testing the cost-effectiveness of the Des Moines–Boston shipping route, we try shipping 1 bathtub from Des Moines to Boston. This is 1 *more* unit than we *were* sending between the two cities, so place a plus sign in the box. However, if we ship 1 more unit than before from Des Moines to Boston, we end up sending 101 bathtubs out of the Des Moines factory. Because the Des Moines factory's capacity is only 100 units, we must ship 1 bathtub less from Des Moines to Albuquerque. This change prevents us from violating the capacity constraint.

To indicate that we have reduced the Des Moines–Albuquerque shipment, place a minus sign in its box. As you continue along the closed path, notice that we are no longer meeting our Albuquerque warehouse requirement for 300 units. In fact, if we reduce the Des Moines–Albuquerque shipment to 99 units, we must increase the Evansville–Albuquerque load by 1 unit, to 201 bathtubs. Therefore, place a plus sign in that box to indicate the increase. You may also observe that those squares in which we turn a corner (and only those squares) will have plus or minus signs.

Figure C.5

Stepping-Stone Evaluation of Alternative Routes for Arizona Plumbing

Evaluation of Des Moines to Boston square



Result of proposed shift in allocation =  $1 \times \$4 - 1 \times \$5 + 1 \times \$8 - 1 \times \$4 = +\$3$

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	100 \$5	Start \$4	\$3	100
(E) Evansville	200 \$8	100 \$4	\$3	300
(F) Fort Lauderdale	\$9	100 \$7	200 \$5	300
Warehouse requirement	300	200	200	700

Finally, note that if we assign 201 bathtubs to the Evansville–Albuquerque route, then we must reduce the Evansville–Boston route by 1 unit, to 99 bathtubs, to maintain the Evansville factory’s capacity constraint of 300 units. To account for this reduction, we thus insert a minus sign in the Evansville–Boston box. By so doing, we have balanced supply limitations among all four routes on the closed path.

**Step 4.** Compute an improvement index for the Des Moines–Boston route by adding unit costs in squares with plus signs and subtracting costs in squares with minus signs.

Des Moines–Boston index =  $\$4 - \$5 + \$8 - \$4 = +\$3$

This means that for every bathtub shipped via the Des Moines–Boston route, total transportation costs will increase by \$3 over their current level.

Let us now examine the unused Des Moines–Cleveland route, which is slightly more difficult to trace with a closed path (see Figure C.6). Again, notice that we turn each corner along the path only at squares on the existing route. Our path, for example, can go through the Evansville–Cleveland box but cannot turn a corner; thus we cannot place a plus or minus sign there. We may use occupied squares only as stepping-stones:

Des Moines–Cleveland index =  $\$3 - \$5 + \$8 - \$4 + \$7 - \$5 = +\$4$

Figure C.6

Testing Des Moines to Cleveland

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	100 \$5	\$4	Start \$3	100
(E) Evansville	200 \$8	100 \$4	\$3	300
(F) Fort Lauderdale	\$9	100 \$7	200 \$5	300
Warehouse requirement	300	200	200	700

Again, opening this route fails to lower our total shipping costs.

Two other routes can be evaluated in a similar fashion:

$$\text{Evansville–Cleveland index} = \$3 - \$4 + \$7 - \$5 = +\$1$$

$$(\text{Closed path} = EC - EB + FB - FC)$$

$$\text{Fort Lauderdale–Albuquerque index} = \$9 - \$7 + \$4 - \$8 = -\$2$$

$$(\text{Closed path} = FA - FB + EB - EA)$$

**INSIGHT** ► Because this last index is negative, we can realize cost savings by using the (currently unused) Fort Lauderdale–Albuquerque route.

**LEARNING EXERCISE** ► What would happen to total cost if Arizona used the shipping route from Des Moines to Cleveland? [Answer: Total cost of the current solution would increase by \$400.]

**RELATED PROBLEMS** ► C.1c, C.3c, C.4–C.11 (C.12, C.13 are available in [MyOMLab](#))

**EXCEL OM** Data File [ModCEXC3.xls](#) can be found in [MyOMLab](#).

In Example C3, we see that a better solution is indeed possible because we can calculate a negative improvement index on one of our unused routes. *Each negative index represents the amount by which total transportation costs could be decreased if one unit was shipped by the source–destination combination.* The next step, then, is to choose that route (unused square) with the *largest* negative improvement index. We can then ship the maximum allowable number of units on that route and reduce the total cost accordingly.

What is the maximum quantity that can be shipped on our new money-saving route? That quantity is found by referring to the closed path of plus signs and minus signs drawn for the route and then selecting the *smallest number found in the squares containing minus signs*. To obtain a new solution, we add this number to all squares on the closed path with plus signs and subtract it from all squares on the path to which we have assigned minus signs.

One iteration of the stepping-stone method is now complete. Again, of course, we must test to see if the solution is optimal or whether we can make any further improvements. We do this by evaluating each unused square, as previously described. Example C4 continues our effort to help Arizona Plumbing arrive at a final solution.

## Example C4

### IMPROVEMENT INDICES

Arizona Plumbing wants to continue solving the problem.

**APPROACH** ► Use the improvement indices calculated in Example C3. We found in Example C3 that the largest (and only) negative index is on the Fort Lauderdale–Albuquerque route (which is the route depicted in Figure C.7).

**SOLUTION** ► The maximum quantity that may be shipped on the newly opened route, Fort Lauderdale–Albuquerque (FA), is the smallest number found in squares containing minus signs—in this case, 100 units. Why 100 units? Because the total cost decreases by \$2 per unit shipped, we know we would like to ship the maximum possible number of units. Previous stepping-stone calculations indicate that each unit shipped over the FA route results in an increase of 1 unit shipped from Evansville (E) to Boston (B) and a decrease of 1 unit in amounts shipped both from F to B (now 100 units) and from E to A (now 200 units). Hence, the maximum we can ship over the FA route is 100 units. This solution results in zero units being shipped from F to B. Now we take the following four steps:

1. Add 100 units (to the zero currently being shipped) on route FA.
2. Subtract 100 from route FB, leaving zero in that square (though still balancing the row total for F).
3. Add 100 to route EB, yielding 200.
4. Finally, subtract 100 from route EA, leaving 100 units shipped.

Note that the new numbers still produce the correct row and column totals as required. The new solution is shown in Figure C.8.

Total shipping cost has been reduced by  $(100 \text{ units}) \times (\$2 \text{ saved per unit}) = \$200$  and is now \$4,000. This cost figure, of course, can also be derived by multiplying the cost of

Figure C.7

Transportation Table:  
Route FA

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	100 \$5	\$4	\$3	100
(E) Evansville	200 \$8	100 \$4	\$3	300
(F) Fort Lauderdale	\$9	100 \$7	200 \$5	300
Warehouse demand	300	200	200	700

**STUDENT TIP** ◀

FA has a negative index:  
FA (+9) to FB (-7) to EB (+4)  
to EA (-8) = -\$2

Figure C.8

Solution at Next Iteration  
(Still Not Optimal)

From \ To	(A) Albuquerque	(B) Boston	(C) Cleveland	Factory capacity
(D) Des Moines	100 \$5	\$4	\$3	100
(E) Evansville	100 \$8	200 \$4	\$3	300
(F) Fort Lauderdale	100 \$9	\$7	200 \$5	300
Warehouse demand	300	200	200	700

shipping each unit by the number of units transported on its respective route, namely:  
 $100(\$5) + 100(\$8) + 200(\$4) + 100(\$9) + 200(\$5) = \$4,000$ .

**INSIGHT** ▶ Looking carefully at Figure C.8, however, you can see that it, too, is not yet optimal. Route EC (Evansville–Cleveland) has a negative cost improvement index of -\$1. Closed path = EC - EA + FA - FC.

**LEARNING EXERCISE** ▶ Find the final solution for this route on your own. [Answer: Programs C.1 and C.2, at the end of this module, provide an Excel OM solution.]

**RELATED PROBLEMS** ▶ C.4–C.11 (C.12–C.13 are available in MyOMLab)

## Special Issues in Modeling

### Demand Not Equal to Supply

A common situation in real-world problems is the case in which total demand is not equal to total supply. We can easily handle these so-called unbalanced problems with the solution procedures that we have just discussed by introducing *dummy sources* or *dummy destinations*. If total supply is greater than total demand, we make demand exactly equal the surplus by creating a dummy destination. Conversely, if total demand is greater than total supply, we introduce a dummy source (factory) with a supply equal to the excess of demand. Because these units will not in fact be shipped, we assign cost coefficients of zero to each square on the dummy location. In each case, then, the cost is zero.

**LO C.3** Balance a transportation problem

### Degeneracy

To apply the stepping-stone method to a transportation problem, we must observe a rule about the number of shipping routes being used: *The number of occupied squares in any solution (initial or later) must be equal to the number of rows in the table plus the number of columns minus 1.* Solutions that do not satisfy this rule are called *degenerate*.

**LO C.4** Deal with a problem that has degeneracy

**Degeneracy**

An occurrence in transportation models in which too few squares or shipping routes are being used, so that tracing a closed path for each unused square becomes impossible.

**Degeneracy** occurs when too few squares or shipping routes are being used. As a result, it becomes impossible to trace a closed path for one or more unused squares. The Arizona Plumbing problem we just examined was *not* degenerate, as it had 5 assigned routes (3 rows or factories + 3 columns or warehouses – 1).

To handle degenerate problems, we must artificially create an occupied cell: That is, we place a zero or a *very* small amount (representing a fake shipment) in one of the unused squares and *then treat that square as if it were occupied*. The chosen square must be in such a position as to allow all stepping-stone paths to be closed.

## Summary

The transportation model, a form of linear programming, is used to help find the least-cost solutions to system-wide shipping problems. The northwest-corner method (which begins in the upper-left corner of the transportation table) or the intuitive lowest-cost method may be used for finding an initial feasible solution. The stepping-stone algorithm is then used for finding optimal solutions. Unbalanced problems are those in which the total

demand and total supply are not equal. Degeneracy refers to the case in which the number of rows + the number of columns – 1 is not equal to the number of occupied squares. The transportation model approach is one of the four location models described earlier in Chapter 8 and is one of the two aggregate planning models discussed in Chapter 13. **Additional solution techniques are presented in Tutorial 4 in MyOMLab.**

### Key Terms

Transportation modeling (p. 730)  
Northwest-corner rule (p. 732)

Intuitive method (p. 733)  
Stepping-stone method (p. 734)

Degeneracy (p. 738)

### Discussion Questions

1. What are the three information needs of the transportation model?
2. What are the steps in the intuitive lowest-cost method?
3. Identify the three “steps” in the northwest-corner rule.
4. How do you know when an optimal solution has been reached?
5. Which starting technique generally gives a better initial solution, and why?
6. The more sources and destinations there are for a transportation problem, the smaller the percentage of all cells that will be used in the optimal solution. Explain.
7. All of the transportation examples appear to apply to long distances. Is it possible for the transportation model to apply on a much smaller scale, for example, within the departments of a store or the offices of a building? Discuss.
8. Develop a *northeast*-corner rule and explain how it would work. Set up an initial solution for the Arizona Plumbing problem analyzed in Example C1.
9. What is meant by an unbalanced transportation problem, and how would you balance it?
10. How many occupied cells must all solutions use?
11. Explain the significance of a negative improvement index in a transportation-minimizing problem.
12. How can the transportation method address production costs in addition to transportation costs?
13. Explain what is meant by the term *degeneracy* within the context of transportation modeling.

### Using Software to Solve Transportation Problems

Excel, Excel OM, and POM for Windows may all be used to solve transportation problems. Excel uses Solver, which requires that you enter your own constraints. Excel OM also uses Solver but is prestructured so that you need enter only the actual data. POM for Windows similarly requires that only demand data, supply data, and shipping costs be entered.

#### **X USING EXCEL OM**

Excel OM’s Transportation module uses Excel’s built-in Solver routine to find optimal solutions to transportation problems. Program C.1 illustrates the input data (from Arizona Plumbing) and total-cost formulas. In Excel 2007, 2010, and 2013 **Solver** is in the **Analysis** section of the **Data** tab. Be certain that the solving method is “Simplex LP.” The output appears in Program C.2.



In Excel 2007, 2010, and 2013, Solver is in the Analysis section of the Data tab. In the prior Excel version or on a Mac with Excel 2011, Solver is on the Tools menu. If Solver is not available, please visit [www.pearsonhighered.com/weiss](http://www.pearsonhighered.com/weiss).

Our target cell is the total cost cell (B21), which we wish to minimize by changing the shipment cells (B16 through D18). The constraints ensure that the number shipped is equal to the number demanded and that we don't ship more units than we have on hand.

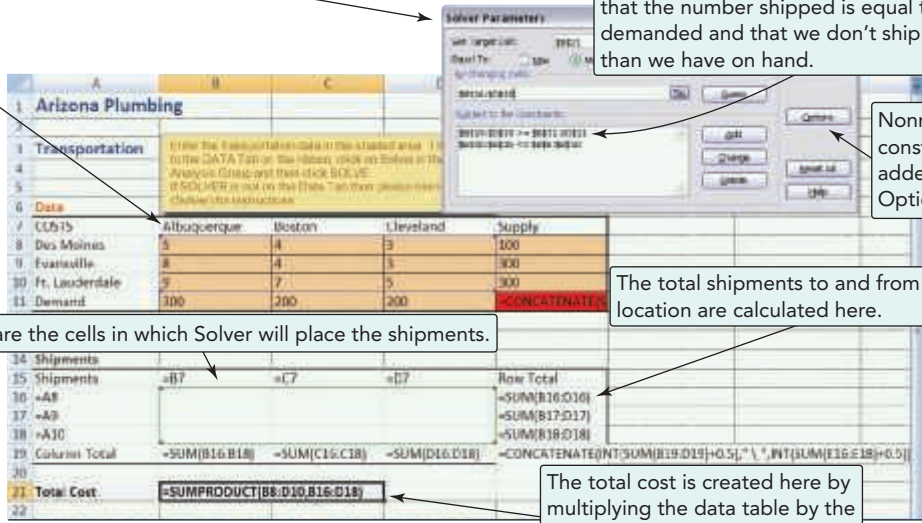
Enter the origin and destination names, the shipping costs, and the total supply and demand figures.

These are the cells in which Solver will place the shipments.

The total shipments to and from each location are calculated here.

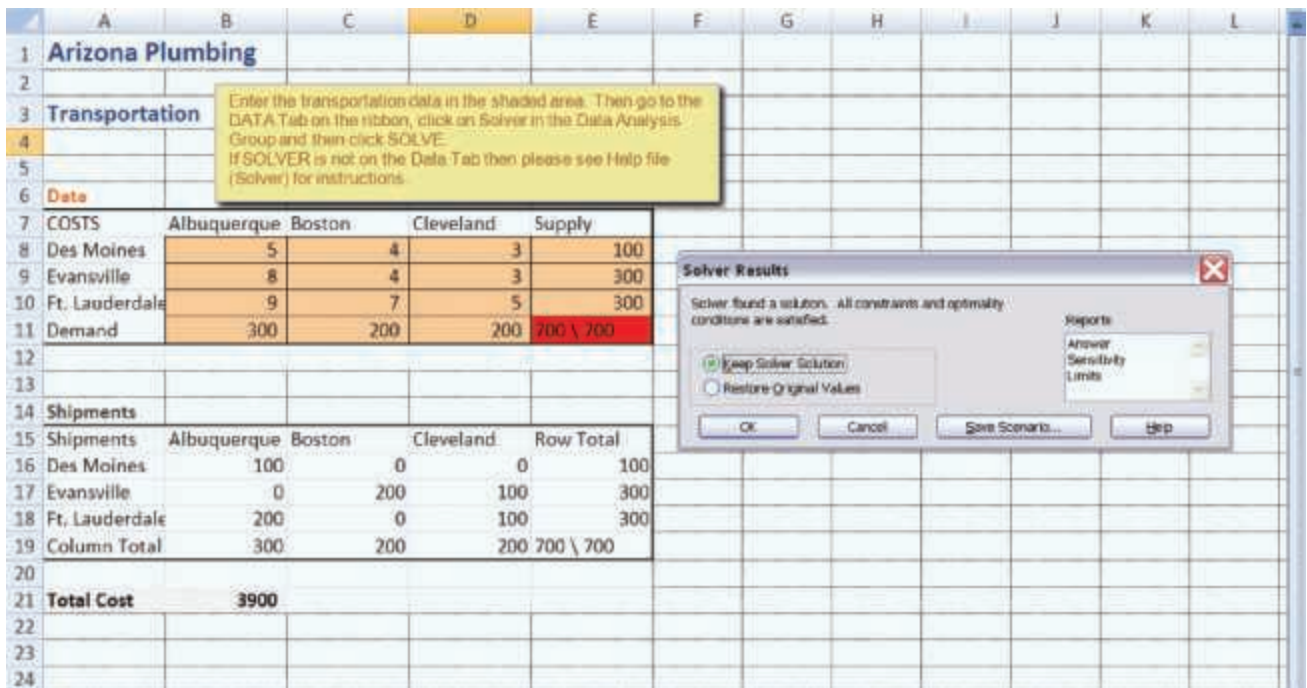
Nonnegativity constraints have been added through the Options button.

The total cost is created here by multiplying the data table by the shipment table using the SUMPRODUCT function.



Program C.1

Excel OM Input Screen and Formulas, Using Arizona Plumbing Data



Program C.2

Output from Excel OM with Optimal Solution to Arizona Plumbing Problem

■ USING POM FOR WINDOWS

The POM for Windows Transportation module can solve both maximization and minimization problems by a variety of methods. Input data are the demand data, supply data, and unit shipping costs. See Appendix IV for further details.

Solved Problems Virtual Office Hours help is available in [MyOMLab](#).

**SOLVED PROBLEM C.1**

Williams Auto Top Carriers currently maintains plants in Atlanta and Tulsa to supply auto top carriers to distribution centers in Los Angeles and New York. Because of expanding demand, Williams has decided to open a third plant and has

narrowed the choice to one of two cities—New Orleans and Houston. Table C.3 provides pertinent production and distribution costs as well as plant capacities and distribution demands.

Which of the new locations, in combination with the existing plants and distribution centers, yields a lower cost for the firm?

**TABLE C.3**

**Production Costs, Distribution Costs, Plant Capabilities, and Market Demands for Williams Auto Top Carriers**

FROM PLANTS	TO DISTRIBUTION CENTERS		NORMAL PRODUCTION	UNIT PRODUCTION COST
	LOS ANGELES	NEW YORK		
Existing plants				
Atlanta	\$8	\$5	600	\$6
Tulsa	\$4	\$7	900	\$5
Proposed locations				
New Orleans	\$5	\$6	500	\$4 (anticipated)
Houston	\$4	\$6 <sup>a</sup>	500	\$3 (anticipated)
Forecast demand	800	1,200	2,000	

<sup>a</sup>Indicates distribution cost (shipping, handling, storage) will be \$6 per carrier between Houston and New York.

**SOLUTION**

To answer this question, we must solve two transportation problems, one for each combination. We will recommend the location that yields a lower total cost of distribution and production in combination with the existing system.

We begin by setting up a transportation table that represents the opening of a third plant in New Orleans (see Figure C.9). Then we use the northwest-corner method to find an initial solution. The total cost of this first solution is \$23,600. Note that the cost of each individual “plant-to-distribution-center” route is found by adding the distribution costs (in the body of Table C.3) to the respective unit production costs (in the right-hand column of Table C.3). Thus, the total production-plus-shipping cost of one auto top carrier from Atlanta to Los Angeles is \$14 (\$8 for shipping plus \$6 for production).

$$\begin{aligned}
 \text{Total cost} &= (600 \text{ units} \times \$14) + (200 \text{ units} \times \$9) \\
 &\quad + (700 \text{ units} \times \$12) + (500 \text{ units} \times \$10) \\
 &= \$8,400 + \$1,800 + \$8,400 + \$5,000 \\
 &= \$23,600
 \end{aligned}$$

From \ To	Los Angeles	New York	Production capacity
Atlanta	\$14 600	\$11	600
Tulsa	\$9 200	\$12 700	900
New Orleans	\$9	\$10 500	500
Demand	800	1,200	2,000

Figure C.9

Initial Williams Transportation Table for New Orleans

Is this initial solution (in Figure C.9) optimal? We can use the stepping-stone method to test it and compute improvement indices for unused routes:

Improvement index for Atlanta–New York route:

$$\begin{aligned}
 &= +\$11 \text{ (Atlanta–New York)} - \$14 \text{ (Atlanta–Los Angeles)} \\
 &\quad + \$9 \text{ (Tulsa–Los Angeles)} - \$12 \text{ (Tulsa–New York)} \\
 &= -\$6
 \end{aligned}$$

Improvement index for New Orleans–Los Angeles route:

$$\begin{aligned}
 &= +\$9 \text{ (New Orleans–Los Angeles)} \\
 &\quad - \$10 \text{ (New Orleans–New York)} \\
 &\quad + \$12 \text{ (Tulsa–New York)} \\
 &\quad - \$9 \text{ (Tulsa–Los Angeles)} \\
 &= \$2
 \end{aligned}$$

Because the firm can save \$6 for every unit shipped from Atlanta to New York, it will want to improve the initial solution and send as many units as possible (600, in this case) on this currently unused route (see Figure C.10). You may also

From \ To	Los Angeles	New York	Production capacity
Atlanta	\$14 600	\$11	600
Tulsa	\$9 800	\$12 100	900
New Orleans	\$9	\$10 500	500
Demand	800	1,200	2,000

Figure C.10

Improved Transportation Table for Williams

want to confirm that the total cost is now \$20,000, a savings of \$3,600 over the initial solution.

Next, we must test the two unused routes to see if their improvement indices are also negative numbers:

Index for Atlanta–Los Angeles:  
 $= \$14 - \$11 + \$12 - \$9 = \$6$

Index for New Orleans–Los Angeles:  
 $= \$9 - \$10 + \$12 - \$9 = \$2$

Because both indices are greater than zero, we have already reached our optimal solution for the New Orleans location. If Williams elects to open the New Orleans plant, the firm’s total production and distribution cost will be \$20,000.

This analysis, however, provides only half the answer to Williams’s problem. The same procedure must still be followed to determine the minimum cost if the new plant is built in Houston. Determining this cost is left as a homework problem. You can help provide complete information and recommend a solution by solving Problem C.7 (on p. 742).

**SOLVED PROBLEM C.2**

In Solved Problem C.1, we examined the Williams Auto Top Carriers problem by using a transportation table. An alternative approach is to structure the same decision analysis using linear programming (LP), which we explained in detail in Business Analytics Module B.

**SOLUTION**

Using the data in Figure C.9 (p. 740), we write the objective function and constraints as follows:

$$\begin{aligned} \text{Minimize total cost} &= \$14X_{\text{Atl,LA}} + \$11X_{\text{Atl,NY}} + \$9X_{\text{Tul,LA}} + \$12X_{\text{Tul,NY}} + \$9X_{\text{NO,LA}} + \$10X_{\text{NO,NY}} \\ \text{Subject to:} \quad X_{\text{Atl,LA}} + X_{\text{Atl,NY}} &\leq 600 && \text{(production capacity at Atlanta)} \\ X_{\text{Tul,LA}} + X_{\text{Tul,NY}} &\leq 900 && \text{(production capacity at Tulsa)} \\ X_{\text{NO,LA}} + X_{\text{NO,NY}} &\leq 500 && \text{(production capacity at New Orleans)} \\ X_{\text{Atl,LA}} + X_{\text{Tul,LA}} + X_{\text{NO,LA}} &\geq 800 && \text{(Los Angeles demand constraint)} \\ X_{\text{Atl,NY}} + X_{\text{Tul,NY}} + X_{\text{NO,NY}} &\geq 1200 && \text{(New York demand constraint)} \end{aligned}$$

**Problems**

Note: **PX** means the problem may be solved with POM for Windows and/or Excel OM.

**Problems C.1–C.3 relate to Developing an Initial Solution**

- **C.1** Find an initial solution to the following transportation problem.

FROM	TO			SUPPLY
	LOS ANGELES	CALGARY	PANAMA CITY	
Mexico City	\$ 6	\$18	\$ 8	100
Detroit	\$17	\$13	\$19	60
Ottawa	\$20	\$10	\$24	40
Demand	50	80	70	

- Use the northwest-corner method. What is its total cost?
- Use the intuitive lowest-cost approach. What is its total cost?
- Using the stepping-stone method, find the optimal solution. Compute the total cost. **PX**

- **C.2** Consider the transportation table at right. Unit costs for each shipping route are in dollars. What is the total cost of the basic feasible solution that the intuitive lowest-cost method would find for this problem? **PX**

Source	Destination					Supply
	A	B	C	D	E	
1	12	8	5	10	4	18
2	6	11	3	7	9	14
Demand	6	8	12	4	2	

- **C.3** Refer to the table that follows.
  - Use the northwest-corner method to find an initial feasible solution. What must you do before beginning the solution steps?
  - Use the intuitive lowest-cost approach to find an initial feasible solution. Is this approach better than the northwest-corner method?
  - Find the optimal solution using the stepping-stone method.

FROM	TO			SUPPLY
	A	B	C	
X	\$10	\$18	\$12	100
Y	\$17	\$13	\$ 9	50
Z	\$20	\$18	\$14	75
Demand	50	80	70	

**Problems C.4–C.13 relate to The Stepping-Stone Method**

• **C.4** Consider the transportation table below. The solution displayed was obtained by performing some iterations of the transportation method on this problem. What is the total cost of the shipping plan that would be obtained by performing *one more iteration* of the stepping-stone method on this problem?

Source	Destination			Supply
	Denver	Yuma	Miami	
Houston	10	\$2	\$8	10
St. Louis	10	\$4	\$5	20
Chicago		\$6	\$3	30
Demand	20	20	20	

•• **C.5** The following table is the result of one or more iterations.

From \ To	1	2	3	Capacity
A	30		5	50
B	10	30	10	30
C	20	30	25	75
Demand	40	60	55	155

- a) Complete the next iteration using the stepping-stone method.
- b) Calculate the “total cost” incurred if your results were to be accepted as the final solution. **Px**

•• **C.6** The three blood banks in Seminole County, Florida, are coordinated through a central office that facilitates blood delivery to four hospitals in the region. The cost to ship a standard container of blood from each bank to each hospital is shown in the table below. Also given are the biweekly number of containers available at each bank and the biweekly number of containers of blood needed at each hospital. How many shipments should be made biweekly from each blood bank to each hospital so that total shipment costs are minimized?



Minerva Studio/Fotolia

FROM	TO				SUPPLY
	HOSP. 1	HOSP. 2	HOSP. 3	HOSP. 4	
Bank 1	\$ 8	\$ 9	\$11	\$16	50
Bank 2	\$12	\$ 7	\$ 5	\$ 8	80
Bank 3	\$14	\$10	\$ 6	\$ 7	120
Demand	90	70	40	50	250



•• **C.7** In Solved Problem C.1 (page 740), Williams Auto Top Carriers proposed opening a new plant in either New Orleans or Houston. Management found that the total system cost (of production plus distribution) would be \$20,000 for the New Orleans site. What would be the total cost if Williams opened a plant in Houston? At which of the two proposed locations (New Orleans or Houston) should Williams open the new facility? **Px**

•• **C.8** The Donna Mosier Clothing Group owns factories in three towns (W, Y, and Z), which distribute to three retail dress shops in three other cities (A, B, and C). The following table summarizes factory availabilities, projected store demands, and unit shipping costs:

From \ To	Dress Shop A	Dress Shop B	Dress Shop C	Factory availability
Factory W	\$4	\$3	\$3	35
Factory Y	\$6	\$7	\$6	50
Factory Z	\$8	\$2	\$5	50
Store demand	30	65	40	135

- a) Complete the analysis, determining the optimal solution for shipping at the Mosier Clothing Group.
- b) How do you know whether it is optimal or not? **Px**

••• **C.9** Captain Borders Corp. manufactures fishing equipment. Currently, the company has a plant in Los Angeles and a plant in New Orleans. William Borders, the firm’s owner, is deciding where to build a new plant—Philadelphia or Seattle. Use the

following table to find the total shipping costs for each potential site. Which should Borders select?

PLANT	WAREHOUSE			CAPACITY
	PITTSBURGH	ST. LOUIS	DENVER	
Los Angeles	\$100	\$75	\$50	150
New Orleans	\$ 80	\$60	\$90	225
Philadelphia	\$ 40	\$50	\$90	350
Seattle	\$110	\$70	\$30	350
Demand	200	100	400	



•• **C.10** Dana Johnson Corp. is considering adding a fourth plant to its three existing facilities in Decatur, Minneapolis, and Carbondale. Both St. Louis and East St. Louis are being considered. Evaluating only the transportation costs per unit as shown in the table, decide which site is best.

TO	FROM EXISTING PLANTS			DEMAND
	DECATUR	MINNEAPOLIS	CARBONDALE	
Blue Earth	\$20	\$17	\$21	250
Ciro	\$25	\$27	\$20	200
Des Moines	\$22	\$25	\$22	350
Capacity	300	200	150	

TO	FROM PROPOSED PLANTS	
	EAST ST. LOUIS	ST. LOUIS
Blue Earth	\$29	\$27
Ciro	\$30	\$28
Des Moines	\$30	\$31
Capacity	150	150



•• **C.11** Using the data from Problem C.10 and the unit production costs in the following table, show which locations yield the lowest cost.

LOCATION	PRODUCTION COSTS (\$)
Decatur	\$50
Minneapolis	60
Carbondale	70
East St. Louis	40
St. Louis	50



Additional problems **C.12–C.13** are available in MyOMLab.

Problems **C.14–C.18** relate to Special Issues in Modeling

•• **C.14** Allen Air Conditioning manufactures room air conditioners at plants in Houston, Phoenix, and Memphis. These are sent to regional distributors in Dallas, Atlanta, and Denver. The shipping costs vary, and the company would like to find the least-cost way to meet the demands at each of the distribution centers. Dallas needs to receive 800 air conditioners per month, Atlanta needs 600, and Denver needs 200. Houston has 850 air conditioners available each month, Phoenix has 650, and Memphis has 300. The shipping cost per unit from Houston to Dallas is \$8, to Atlanta \$12, and to Denver \$10. The cost per unit from Phoenix to Dallas is \$10, to Atlanta \$14, and to Denver \$9. The cost per unit from Memphis to Dallas is \$11, to Atlanta \$8, and to Denver \$12. How many units should owner Stephen Allen ship from each plant to each regional distribution center? What is the total transportation cost? (Note that a “dummy” destination is needed to balance the problem.)

•• **C.15** For the following Gregory Bier Corp. data, find the starting solution and initial cost using the northwest-corner method. What must you do to balance this problem?

FROM	TO				SUPPLY
	W	X	Y	Z	
A	\$132	\$116	\$250	\$110	220
B	\$220	\$230	\$180	\$178	300
C	\$152	\$173	\$196	\$164	435
Demand	160	120	200	230	



Additional problems **C.16–C.18** are available in MyOMLab.

## CASE STUDY

### Custom Vans, Inc.

Custom Vans, Inc., specializes in converting standard vans into campers. Depending on the amount of work and customizing to be done, the customizing can cost from less than \$1,000 to more than \$5,000. In less than 4 years, Tony Rizzo was able to expand his small operation in Gary, Indiana, to other major outlets in Chicago, Milwaukee, Minneapolis, and Detroit.

Innovation was the major factor in Tony’s success in converting a small van shop into one of the largest and most profitable custom van operations in the Midwest. Tony seemed to have a special ability to design and develop unique features and devices that were always in high demand by van owners. An example was Shower-Rific, which was developed by Tony only 6 months after Custom Vans, Inc., was started. These small showers were completely self-contained, and they could be placed in almost any type of van and in a number of different locations within a van. Shower-Rific was made of fiberglass, and contained towel racks,

built-in soap and shampoo holders, and a unique plastic door. Each Shower-Rific took 2 gallons of fiberglass and 3 hours of labor to manufacture.

Most of the Shower-Rifics were manufactured in Gary in the same warehouse where Custom Vans, Inc., was founded. The manufacturing plant in Gary could produce 300 Shower-Rifics in a month, but this capacity never seemed to be enough. Custom Van shops in all locations were complaining about not getting enough Shower-Rifics, and because Minneapolis was farther away from Gary than the other locations, Tony was always inclined to ship Shower-Rifics to the other locations before Minneapolis. This infuriated the manager of Custom Vans at Minneapolis, and after many heated discussions, Tony decided to start another manufacturing plant for Shower-Rifics at Fort Wayne, Indiana. The manufacturing plant at Fort Wayne could produce 150 Shower-Rifics per month.

The manufacturing plant at Fort Wayne was still not able to meet current demand for Shower-Rifics, and Tony knew that the demand for his unique camper shower would grow rapidly in the next year. After consulting with his lawyer and banker, Tony concluded that he should open two new manufacturing plants as soon as possible. Each plant would have the same capacity as the Fort Wayne manufacturing plant. An initial investigation into possible manufacturing locations was made, and Tony decided that the two new plants should be located in Detroit, Michigan; Rockford, Illinois; or Madison, Wisconsin. Tony knew that selecting the best location for the two new manufacturing plants would be difficult. Transportation costs and demands for the various locations would be important considerations.

The Chicago shop was managed by Bill Burch. This shop was one of the first established by Tony, and it continued to outperform the other locations. The manufacturing plant at Gary was supplying 200 Shower-Rifics each month, although Bill knew that the demand for the showers in Chicago was 300 units. The transportation cost per unit from Gary was \$10, and although the transportation cost from Fort Wayne was double that amount, Bill was always pleading with Tony to get an additional 50 units from the Fort Wayne manufacturer. The two additional manufacturing plants would certainly be able to supply Bill with the additional 100 showers he needed. The transportation costs would, of course, vary, depending on which two locations Tony picked. The transportation cost per shower would be \$30 from Detroit, \$5 from Rockford, and \$10 from Madison.

Wilma Jackson, manager of the Custom Van shop in Milwaukee, was the most upset about not getting an adequate supply of showers. She had a demand for 100 units, and at the present time, she was only getting half of this demand from the Fort Wayne manufacturing plant. She could not understand why Tony didn't ship her all 100 units from Gary. The transportation cost per unit from Gary was only \$20, while the transportation cost from Fort Wayne was \$30. Wilma was hoping that Tony would select Madison for one of the manufacturing locations. She would be able to get all the showers needed, and the transportation cost per unit would only be \$5. If not in Madison, a new plant in Rockford would be able to supply her total needs, but the transportation cost per unit would be twice as much as it would be from Madison. Because the transportation cost per unit from Detroit would be \$40, Wilma speculated that even if Detroit became one of the new plants, she would not be getting any units from Detroit.

Custom Vans, Inc., of Minneapolis was managed by Tom Poanski. He was getting 100 showers from the Gary plant. Demand was 150 units. Tom faced the highest transportation costs of all locations. The transportation cost from Gary was \$40 per unit. It would cost \$10 more if showers were sent from the Fort Wayne location. Tom was hoping that Detroit would not be one of the new plants, as the transportation cost would be \$60 per unit. Rockford and Madison would have a cost of \$30 and \$25, respectively, to ship one shower to Minneapolis.

The Detroit shop's position was similar to Milwaukee's—only getting half of the demand each month. The 100 units that Detroit did receive came directly from the Fort Wayne plant. The transportation cost was only \$15 per unit from Fort Wayne, while it was

\$25 from Gary. Dick Lopez, manager of Custom Vans, Inc., of Detroit, placed the probability of having one of the new plants in Detroit fairly high. The factory would be located across town, and the transportation cost would be only \$5 per unit. He could get 150 showers from the new plant in Detroit and the other 50 showers from Fort Wayne. Even if Detroit was not selected, the other two locations were not intolerable. Rockford had a transportation cost per unit of \$35, and Madison had a transportation cost of \$40.

Tony pondered the dilemma of locating the two new plants for several weeks before deciding to call a meeting of all the managers of the van shops. The decision was complicated, but the objective was clear—to minimize total costs. The meeting was held in Gary, and everyone was present except Wilma.

*Tony:* Thank you for coming. As you know, I have decided to open two new plants at Rockford, Madison, or Detroit. The two locations, of course, will change our shipping practices, and I sincerely hope that they will supply you with the Shower-Rifics that you have been wanting. I know you could have sold more units, and I want you to know that I am sorry for this situation.

*Dick:* Tony, I have given this situation a lot of consideration, and I feel strongly that at least one of the new plants should be located in Detroit. As you know, I am now only getting half of the showers that I need. My brother, Leon, is very interested in running the plant, and I know he would do a good job.

*Tom:* Dick, I am sure that Leon could do a good job, and I know how difficult it has been since the recent layoffs by the auto industry. Nevertheless, we should be considering total costs and not personalities. I believe that the new plants should be located in Madison and Rockford. I am farther away from the other plants than any other shop, and these locations would significantly reduce transportation costs.

*Dick:* That may be true, but there are other factors. Detroit has one of the largest suppliers of fiberglass, and I have checked prices. A new plant in Detroit would be able to purchase fiberglass for \$2 per gallon less than any of the other existing or proposed plants.

*Tom:* At Madison, we have an excellent labor force. This is due primarily to the large number of students attending the University of Wisconsin. These students are hard workers, and they will work for \$1 less per hour than the other locations that we are considering.

*Bill:* Calm down, you two. It is obvious that we will not be able to satisfy everyone in locating the new plants. Therefore, I would like to suggest that we vote on the two best locations.

*Tony:* I don't think that voting would be a good idea. Wilma was not able to attend, and we should be looking at all of these factors together in some type of logical fashion.

### Discussion Question

Where would you locate the two new plants? Why?

*Source:* From *Quantitative Analysis for Management*, B. Render, R. M. Stair, M. Hanna, and T. Hale. 12th ed. Copyright © 2015. Reprinted by permission of Pearson Publishing, Upper Saddle River, NJ.

- **Additional Case Study:** Visit [MyOMLab](#) for this free case study:
- **Consolidated Bottling (B):** This case involves determining where to add bottling capacity.

# Module C *Rapid Review*

MyOMLab

## Main Heading Review Material

### TRANSPORTATION MODELING (pp. 730–731)

The transportation models described in this module prove useful when considering alternative facility locations *within the framework of an existing distribution system*. The choice of a new location depends on which will yield the minimum cost for the entire system.

- **Transportation modeling**—An iterative procedure for solving problems that involves minimizing the cost of shipping products from a series of sources to a series of destinations.

*Origin points* (or *sources*) can be factories, warehouses, car rental agencies, or any other points from which goods are shipped.

*Destinations* are any points that receive goods.

To use the transportation model, we need to know the following:

1. The origin points and the capacity or supply per period at each.
2. The destination points and the demand per period at each.
3. The cost of shipping one unit from each origin to each destination.

The transportation model is a type of linear programming model.

A *transportation matrix* summarizes all relevant data and keeps track of algorithm computations. Shipping costs from each origin to each destination are contained in the appropriate cross-referenced box.

FROM \ TO	DESTINATION 1	DESTINATION 2	DESTINATION 3	CAPACITY
Source A				
Source B				
Source C				
Demand				

Concept Questions:  
1.1–1.4

### DEVELOPING AN INITIAL SOLUTION (pp. 732–734)

Two methods for establishing an initial feasible solution to the problem are the northwest-corner rule and the intuitive lowest-cost method.

- **Northwest-corner rule**—A procedure in the transportation model where one starts at the upper-left-hand cell of a table (the northwest corner) and systematically allocates units to shipping routes.

The northwest-corner rule requires that we:

1. Exhaust the supply (origin capacity) of each row before moving down to the next row.
2. Exhaust the demand requirements of each column before moving to the next column to the right.
3. Check to ensure that all supplies and demands are met.

The northwest-corner rule is easy to use and generates a feasible solution, but it totally ignores costs and therefore should be considered only as a starting position.

- **Intuitive method**—A cost-based approach to finding an initial solution to a transportation problem.

The intuitive method uses the following steps:

1. Identify the cell with the lowest cost. Break any ties for the lowest cost arbitrarily.
2. Allocate as many units as possible to that cell, without exceeding the supply or demand. Then cross out that row or column (or both) that is exhausted by this assignment.
3. Find the cell with the lowest cost from the remaining (not crossed out) cells.
4. Repeat Steps 2 and 3 until all units have been allocated.

Concept Questions:  
2.1–2.4  
Problems: C.1–C.3, C.15

### THE STEPPING-STONE METHOD (pp. 734–737)

- **Stepping-stone method**—An iterative technique for moving from an initial feasible solution to an optimal solution in the transportation method.

The stepping-stone method is used to evaluate the cost-effectiveness of shipping goods via transportation routes not currently in the solution. When applying it, we test each unused cell, or square, in the transportation table by asking: What would happen to total shipping costs if one unit of the product were tentatively shipped on an unused route? We conduct the test as follows:

1. Select any unused square to evaluate.
2. Beginning at this square, trace a closed path back to the original square via squares that are currently being used (only horizontal and vertical moves are permissible). You may, however, step over either an empty or an occupied square.

Concept Questions:  
3.1–3.4

Problems: C.4–C.13

Virtual Office Hours  
for Solved Problem: C.1

Main Heading	Review Material	
	<p>3. Beginning with a plus (+) sign at the unused square, place alternative minus signs and plus signs on each corner square of the closed path just traced.</p> <p>4. Calculate an improvement index by first adding the unit-cost figures found in each square containing a plus sign and then subtracting the unit costs in each square containing a minus sign.</p> <p>5. Repeat Steps 1 through 4 until you have calculated an improvement index for all unused squares. If all indices computed are <i>greater than or equal to zero</i>, you have reached an optimal solution. If not, the current solution can be improved further to decrease total shipping costs.</p> <p><i>Each negative index represents the amount by which total transportation costs could be decreased if one unit was shipped by the source–destination combination. The next step, then, is to choose that route (unused square) with the largest negative improvement index. We can then ship the maximum allowable number of units on that route and reduce the total cost accordingly. That maximum quantity is found by referring to the closed path of plus signs and minus signs drawn for the route and then selecting the smallest number found in the squares containing minus signs. To obtain a new solution, we add this number to all squares on the closed path with plus signs and subtract it from all squares on the path to which we have assigned minus signs. From this new solution, a new test of unused squares needs to be conducted to see if the new solution is optimal or whether we can make further improvements.</i></p>	
<b>SPECIAL ISSUES IN MODELING</b> (pp. 737–738)	<p><i>Dummy sources</i>—Artificial shipping source points created when total demand is greater than total supply to effect a supply equal to the excess of demand over supply.</p> <p><i>Dummy destinations</i>—Artificial destination points created when the total supply is greater than the total demand; they serve to equalize the total demand and supply.</p> <p>Because units from dummy sources or to dummy destinations will not in fact be shipped, we assign cost coefficients of zero to each square on the dummy location. If you are solving a transportation problem by hand, be careful to decide first whether a dummy source (row) or a dummy destination (column) is needed.</p> <p>When applying the stepping-stone method, <i>the number of occupied squares in any solution (initial or later) must be equal to the number of rows in the table plus the number of columns minus 1</i>. Solutions that do not satisfy this rule are called <i>degenerate</i>.</p> <ul style="list-style-type: none"> <li>▪ <b>Degeneracy</b>—An occurrence in transportation models in which too few squares or shipping routes are being used, so that tracing a closed path for each unused square becomes impossible.</li> </ul> <p>To handle degenerate problems, we must artificially create an occupied cell: That is, we place a zero (representing a fake shipment) into one of the unused squares and <i>then treat that square as if it were occupied</i>. Remember that the chosen square must be in such a position as to allow all stepping-stone paths to be closed.</p>	<p>Concept Questions: 4.1–4.4</p> <p>Problems: C.14–C.18</p> <p>Virtual Office Hours for Solved Problem: C.2</p>

## Self Test

■ **Before taking the self-test**, refer to the learning objectives listed at the beginning of the module and the key terms listed at the end of the module.

**LO C.1** With the transportation technique, the initial solution can be generated in any fashion one chooses. The only restriction(s) is that:

- a) the solution be optimal.
- b) one use the northwest-corner method.
- c) the edge constraints for supply and demand be satisfied.
- d) the solution not be degenerate.
- e) all of the above.

**LO C.2** The purpose of the stepping-stone method is to:

- a) develop the initial solution to a transportation problem.
- b) identify the relevant costs in a transportation problem.
- c) determine whether a given solution is feasible.
- d) assist one in moving from an initial feasible solution to the optimal solution.
- e) overcome the problem of degeneracy.

**LO C.3** The purpose of a *dummy source* or a *dummy destination* in a transportation problem is to:

- a) provide a means of representing a dummy problem.
- b) obtain a balance between total supply and total demand.
- c) prevent the solution from becoming degenerate.
- d) make certain that the total cost does not exceed some specified figure.
- e) change a problem from maximization to minimization.

**LO C.4** If a solution to a transportation problem is degenerate, then:

- a) it will be impossible to evaluate all empty cells without removing the degeneracy.
- b) a dummy row or column must be added.
- c) there will be more than one optimal solution.
- d) the problem has no feasible solution.
- e) increase the cost of each cell by 1.

Answers: LO C.1. c; LO C.2. d; LO C.3. b; LO C.4. a.